Topological Lower Bounds for Decision Trees

Anders Björner
KTH
Stockholm, Sweden

"k-equal problem"

Given $x_1, \dots, x_n \in \mathbb{R}$, and $k \geq 2$, how many comparisons ($x_i \lesseqqgtr x_j$) are needed to decide whether some k of the x_i's are equal: $x_{i_1} = x_{i_2} = \dots = x_{i_k}$?

(I.e., best algorithm & worst case)

① Upper bound: Complexity $\leq 8 \cdot n \log_3\left(\frac{n}{k}\right)$

Construct algorithm with this performance

Say $n = k \cdot 2^m$, $m = \log_2\left(\frac{n}{k}\right)$

$m \cdot 3n$ { Find median using $\leq 3n$ comparisons
Repeat m times for successive subintervals

$2n$ { Compare red elem. with subinterval before & after

$$3nm + 2n \leq 8n\log_3\left(\frac{n}{k}\right)$$

② Lower bound: Complexity $\geq$ const $\cdot\, n \log\left(\frac{n}{k}\right)$

$\to$ Topology and Combinatorics

"Why can we not do this faster?"

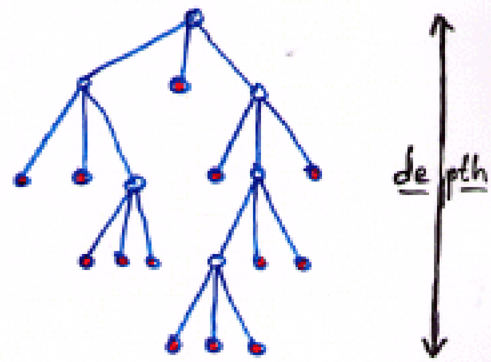
References:

- Björner – Lovász – Yao, 24th STOC, pp. 170 - 177, 1992.
- Björner – Lovász, Journ. AMS $\underline{\underline{7}}$ (1994), pp. 677 – 706.

Geometric view of k-equal problem

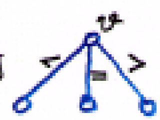
Let $V_{n,k} = \{ x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid$ some k coordinates are equal: $x_{i_1} = \dots = x_{i_k} \}$

- Decide "$x \in V_{n,k}$?" for inputs $x \in \mathbb{R}^n$
- Algorithm = Linear Decision Tree



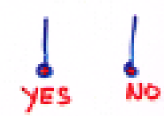
— linear fctn $f_v = a_1 x_1 + \dots + a_n x_n$ at every inner node v

— branching



as $f_v(x) \lesseqgtr 0$

— YES or NO label on each leaf



$\boxed{\text{Complexity} = \text{Depth of best tree} \geq \log_3(\#\text{ leaves})}$

More generally: For $E \subseteq \mathbb{R}^n$, decide
"$x \in E$?" for inputs $x \in \mathbb{R}^n$
via linear decision tree algorithm.

"Component Count Method":

$$\text{Complexity} \geq \log_3 (\# \text{conn. comp. of } E^c = \mathbb{R}^n \setminus E)$$

[Dobkin–Lipton '75, Steele–Yao '82, Ben–Or '83]

"Element Distinctness" (2-equal probl.)

$$\text{Complexity} \geq \log_3 (n!) = \Omega(n \log n)$$

(E^c = complement of braid arrangement $\Rightarrow$ $n!$ chambers)
$\overset{\shortparallel}{V^c_{n,2}}$

Note: For $k > 2$, $V^c_{n,k}$ is connected,
so CC method useless!

<u>Let</u> O = open convex set $\subseteq \mathbb{R}^n$

E = closed polyhedron $\subseteq \mathbb{R}^n$

T = linear dec. tree testing points in O for membership in E

<u>Thm</u> [Bj-Lovász, '94]: $\ell^{-}_{n-i}(T) \geq \beta_i(O \setminus E)$

of $(n-i)$-dim'l NO-leaves

i-th Betti nr.

<u>Coroll.</u>: depth $(T) \geq \log_3(\beta(O \setminus E))$ sum of all Betti nrs.

In particular, for k-equal problem:

complexity $\geq \log(\underbrace{\beta(M_{n,k})}_{\text{How to compute ?}})$

where $M_{n,k} = \mathbb{R}^n \setminus V_{n,k}$

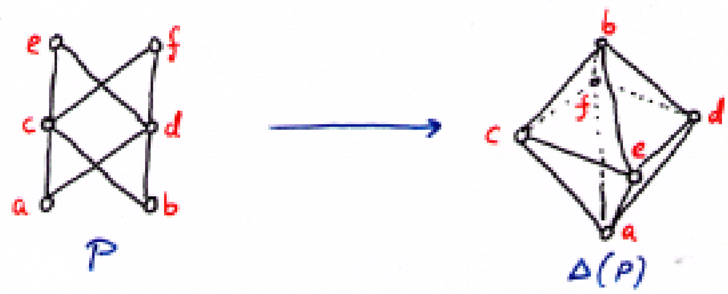
Review of "order complexes".

Poset P (finite partially ordered set)

$\Delta(P) = \{$ chains $x_0 < x_1 < \ldots < x_d$ in $P\}$

simplicial complex

Ex:



Thus, topological concepts apply to posets

Möbius number $\mu(P) = \tilde{\chi}(\Delta(P))$

reduced Euler char.

Interval $(x,y) := \{ z \in P \mid x < z < y \}$

Subspace Arrangements

$\mathcal{A} = \{K_1, \ldots, K_t\}$, K_i linear subspaces in $\mathbb{R}^n$

$L_{\mathcal{A}} = \{ K_{i_1} \cap \ldots \cap K_{i_j} \mid 1 \le i_1 < \ldots < i_j \le t \}$

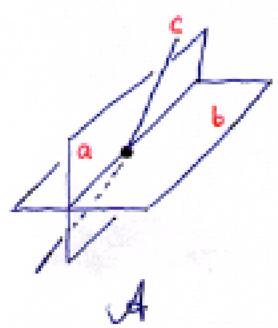
ordered by: $X \le Y \Leftrightarrow X \supseteq Y$

intersection lattice

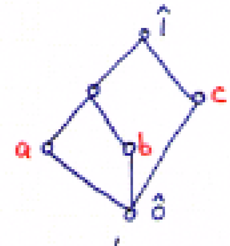
Ex: $\mathcal{A}_{n,k} = \{ x_{i_1} = \ldots = x_{i_k} \mid 1 \le i_1 < \ldots < i_k \le n \}$

"k-equal arrangement"

$k=2$: braid arr't



$\mathcal{A}$

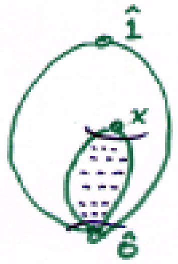


$L_{\mathcal{A}}$

COHOMOLOGY OF COMPLEMENT

<u>Thm</u> (Goresky - MacPherson, 1988)

$$\tilde{H}^i(M_{\mathcal{A}}) \cong \bigoplus_{x \in L_{\mathcal{A}} - \{\hat{0}\}} \tilde{H}_{\mathrm{codim}(x)-2-i}(\hat{0}, x)$$



<u>Proofs</u>: Stratified Morse Theory (G-M)

Spectral Sequence (Vassiliev, Jewell-Orlik-Shapiro)

Homotopy Methods (Ziegler - Zivaljevic)

<u>Comments</u>:

- For $\mathbb{R}$-hyperplane arr't: Specializes to Zaslavsky's formula for $\beta_0(M_{\mathcal{A}}) = \#$ connected components

- For $\mathbb{C}$-hyperplane arr't: Specializes to Orlik-Solomon formula for $\beta_i(M_{\mathcal{A}})$

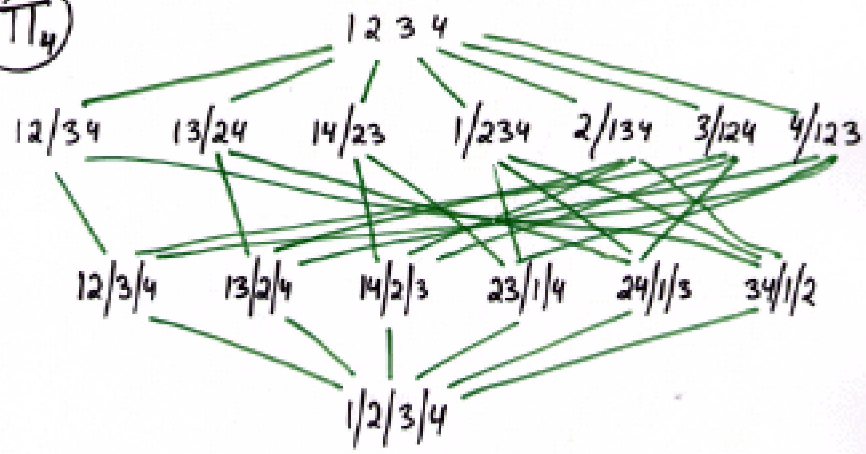
$[n] = \{1, 2, \dots, n\}$

Partition of $[n]$

e.g. $146/28/3/57$

Partition lattice Π_n

Π_4

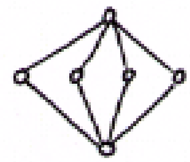


$$\left\{\begin{array}{l}\text{Back to } k\text{-equal arr't } \mathcal{A}_{n,k} \\ \text{Intersection lattice ?}\end{array}\right\}$$

RECALL:

The "k-equal" partition lattice

$$\Pi_{n,k} := \{ \pi \in \Pi_n \mid \text{no blocks of sizes } 2, 3, \ldots, k-1 \}$$

Ex: $\Pi_{n,2} = \Pi_n$, $\Pi_{4,3} =$ 

$$\boxed{\text{FACT: } L_{A_{n,k}} \cong \Pi_{n,k}}$$

Thm [Bj - Welker '95]

- $\tilde{H}_*(\Pi_{n,k})$ is torsion-free
- $\beta^d := \operatorname{rank} \tilde{H}_d(\Pi_{n,k}) \neq 0$

  $\iff d = n-3-t(k-2)$, $1 \leq t \leq \frac{n}{k}$
- $$\beta^{n-3-t(k-2)} = (t-1)! \sum_{0 = i_0 \leq \ldots \leq i_t = n-tk} \prod_{j=0}^{t-1} \binom{n-jk-i_j-1}{k-1} (j+1)^{i_{j+1}-i_j}$$

Now, feed $\tilde{H}_*(\Pi_{n,k})$ information into the Goresky - MacPherson formula to obtain following for the complement $M_{n,k}$ of $\mathcal{A}_{n,k}$ in $\mathbb{R}^n$.

<u>Thm</u> [Bj - Welker '95]

- $\tilde{H}^i(M_{n,k}) \neq 0 \iff i = t(k-2)$
  $$0 \leq t \leq \frac{n}{k}$$
- Formulas for non-zero Betti nrs, e.g.
  $$\beta_*^{k-2}(M_{n,k}) = \sum_{i=k}^{n} \binom{n}{i}\binom{i-1}{k-1} \quad , k \geq 3$$
- $\tilde{H}^*(M_{n,k})$ is torsion-free
- Computation of $\tilde{H}^*(M_{n,k}^{\mathbb{C}})$ - groups

Remark: Möbius function

$$\mu_{n,k} := \begin{cases} \mu(\Pi_{n,k}) = \tilde{\chi}(\Delta(\Pi_{n,k})) & , \text{ if } k \le n \\ 0 & , 1 < n < k \\ 1 & , n = 1 \end{cases}$$

Thm [Bj - Lovász - Yao '92] For fixed $k \geq 2$,

$$\exp\left(\sum_{n \geq 1} \mu_{n,k} \frac{x^n}{n!}\right) = \underbrace{1 + x + \frac{x^2}{2!} + \dots + \frac{x^{k-1}}{(k-1)!}}_{p_k(x) :=}$$

Coroll: Let $\alpha_1, \dots, \alpha_{k-1}$ be roots of $p_k(x)$. Then

$$\mu_{n,k} = -(n-1)! \sum_{i=1}^{k-1} \alpha_i^{-n}$$

Ex: $\underline{\underline{k=3}}$ case $\Rightarrow$ $\alpha_1 = -1+i$, $\alpha_2 = -1-i$ $\Rightarrow$

$$* \quad \mu_{n,3} = -(n-1)!\, 2^{1-n/2} \cos\frac{3\pi n}{4}$$

so $\mu_{n,3} = 0$ for $n = 6, 10, 14, \dots$

Summing Up:

$c(n,k) :=$ linear decision tree complexity of k-equal problem

$$c(n,k) \geq \log_3(\beta(M_{n,k})) \geq \log_3(|\mu_{n,k}|)$$

[used in Bj–Lovász–Yao '92]

[used in Bj–Lovász '94]

$$\boxed{C_1 \cdot n \log_3 \frac{2n}{k} \geq c(n,k) \geq C_2 \cdot n \log_3 \frac{2n}{k}}$$

Elementary

Alg. Topology + Combinatorics

Remark: $C_1 / C_2 \leq 16$

By the same technique :

Given : $x_1, \dots, x_n \in \mathbb{R}$, $k \geq 2$

How many comparisons , linear tests or degree $\leq d$ polynomial tests are needed (best algorithm , worst case) to decide :

(1) Are some k x_i's pairwise unequal ?

Answer : $\Theta(n \log k)$

(2) Is the # of x_i's having any given value divisible by k ?

Answer : $\Theta(n \log \frac{n}{k})$

(3) Is the # of x_i's having any given value either $= 0$ or $\geq k$?

Answer [S. Linnsson] : $\Theta(n \log \frac{n}{k})$

(4) Given $n \times n$ $\mathbb{R}$-matrix (x_{ij}), is there k-matching $(i_1, j_1), \dots, (i_k, j_k)$ such that $x_{i_1 j_1} = \dots = x_{i_k j_k}$?

Answer [A. Yao] : $\Theta(n^2 \log n)$

Variations & Extensions

- Degree d algebraic decision tree

  Replace linear tests by polynomials of degree $\le d$

- Algebraic computation tree

  Also has arithmetic nodes $z \leftarrow x+y$, etc.

- Arithmetic networks

  Directed acyclic graph, several output nodes
  (models parallel computation)

Let E be semialgebraic set $\subseteq \mathbb{R}^n$

$\hat{E}$ one-pt compactification

<u>Thm</u> [Yao '94] $\exists$ const $\lambda_d, \mu_d > 0$ such that

depth of deg-d-algebraic decision tree

$$\geq \lambda_d \log \beta(\hat{E}) - \mu_d n$$

- similar for alg. computation trees
- uses Oleinik-Petrovsky-Milnor-Thom upper bounds for β of semialgebr. sets

<u>Thm</u> [Montaña-Morais-Pardo '96] $\exists$ const $\gamma > 0$ s.t.

$$\text{depth of arithmetic network} \geq \gamma \sqrt{\frac{\log \beta(\hat{E})}{n}}$$

Membership problem for convex polyhedron $E \subseteq \mathbb{R}^n$

[Grigoriev – Karpinski – Vorobjov '94]

Note: E contractible $\Rightarrow \beta(E) = \chi(E) = 1$!

$\Rightarrow$ topological measures useless

Thm: Depth of alg. tree $\geq$ const $\cdot \log f(E)$

$f(E) :=$ # faces of all dimensions

- Uses some diff. geometry for non-smooth pts on boundary

Algebraic trees computing complex roots

[Smale '87, Vassiliev '88 '89]

Task: Given $p(z) = z^n + a_1 z^{n-1} + \ldots + a_{n-1} z + a_n \in \mathbb{C}[z]$

and $\varepsilon > 0$, find the roots of $p(z)$ to within ε.

Let $c(n,\varepsilon) :=$ # leaves of best alg. computation tree

Thm [Vassiliev] $c(n,\varepsilon) = n$, if n = prime-power

something $\leq c(n,\varepsilon) \leq n$, in general

(for ε suff. small > 0)

- Proof uses covering nr [or, Schwarz genus]

  of map $M^{\mathbb{C}}_{n,2} \longrightarrow$ {monic pol's w. distinct roots}

- Finding just one root (within ε) equally hard

Decision trees with $\mathbb{Z}$-inputs

[A. Yao '91]

- Component count method adapted

  $\rightarrow$ count only conn. comp. of measure > 0

  (+ a few more technical details)

- $\Omega(n \log n)$ lower bound for Element Distinctness

- $\Omega(n \log \frac{2n}{k})$ lower bound for k-equal problem (with $\mathbb{Z}$-linear tests)

Algebraic trees in characteristic $p > 0$

[M. Ben-Or '94]

$F = GF(q)$, $\overline{F}$ = alg. closure

- Membership problem for algebraic set $E \subseteq \overline{F}^n$
- Zeta fctn $Z(E,t) := \exp\left(\sum_{s \geq 1} N_s \frac{t^s}{s}\right)$

  where $N_s :=$ #pts of E over $GF(q^s)$
- [Dwork '60] $Z(E,t) = \frac{P(t)}{Q(t)}$ rational !
- Let $\beta(E) := \deg P + \deg Q$
- Thm: Depth of best deg d tree $\geq \lambda_d \log(\beta(E)) - \mu_d n$

  (Milnor-Thom replaced by Bombieri) $\lambda_d, \mu_d > 0$
- k-equal problem $\longrightarrow$ same answer

Final comments:

(1) Can method be generalized to n-valued decision trees?

(YES/NO $\leftrightarrow$ 2-valued) $\quad f: \mathbb{R}^n \to \{1,2,\dots,n\}$

Role of topology of the configuration of n sets $f^{-1}(i)$?

(2) Is there a similar method in Boolean circuit complexity?

$$f: \{0,1\}^n \longrightarrow \{0,1\}$$

Compute $f(x)$ $\longleftrightarrow$ Membership problem

$$x \in f^{-1}(0) \subseteq \{0,1\}^n \subseteq \mathbb{F}_2^n$$

$f^{-1}(0)$: zero-set of ideal (not unique)

$\{0,1\}^n \subseteq \mathbb{F}_2^n$: affine n-space over $GF(2) = \mathbb{F}_2$

1 f monotone $\Rightarrow$ $f^{-1}(0)$ simplicial complex $\to$ topology .... ?

2 Étale cohomology gives Betti numbers .... ?