Achronal Simplexes and Deformations of Higher Dimensional Automata

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Section 1 : Encoding Algebraic Properties of Execution Paths

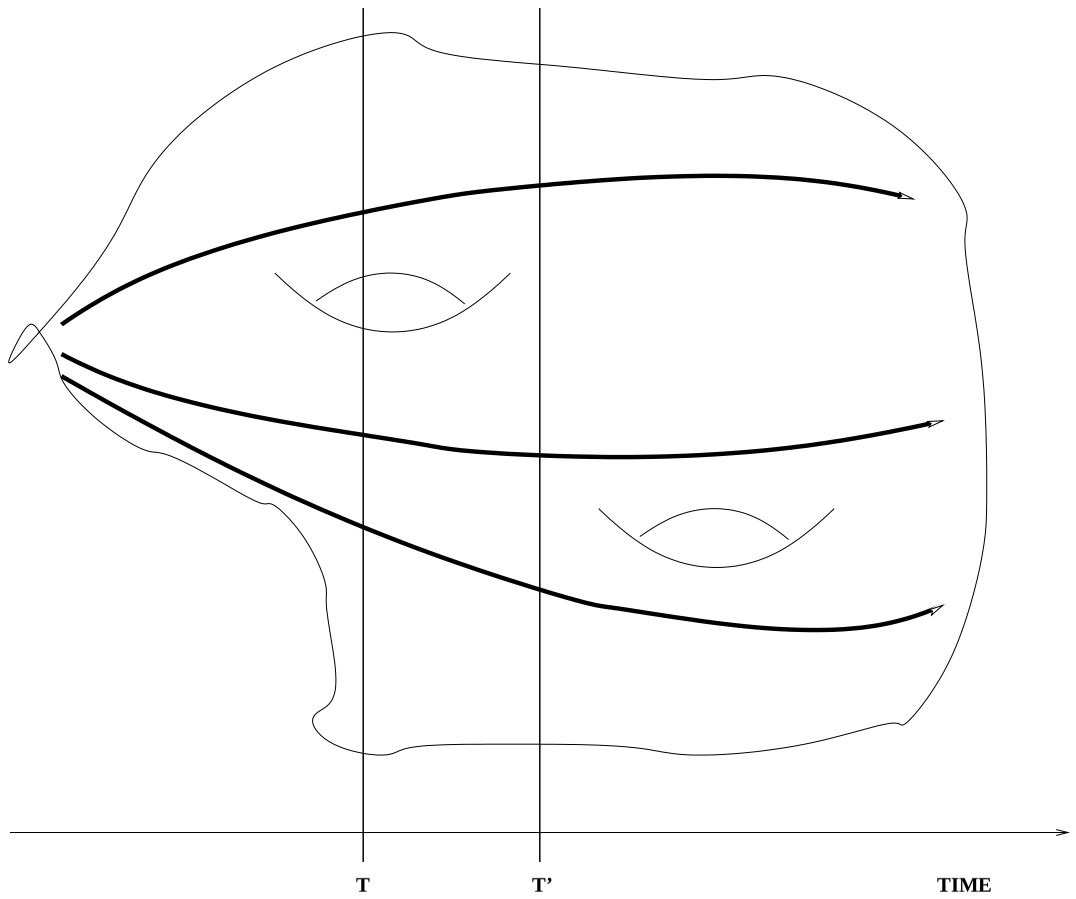


Figure 1: A HDA as a flow of execution paths

Cubical Modeling of HDA

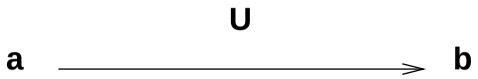


Figure 2: A 1-dimensional transition

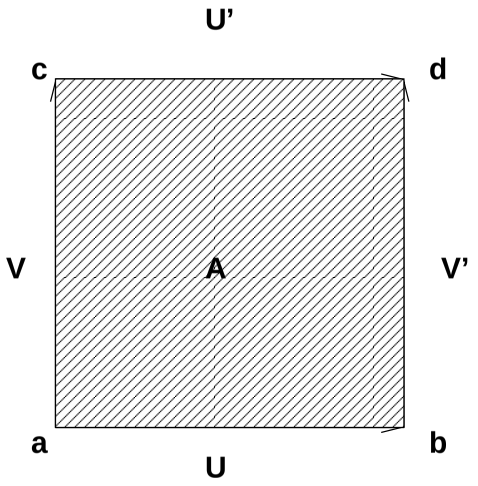


Figure 3: A 2-dimensional transition

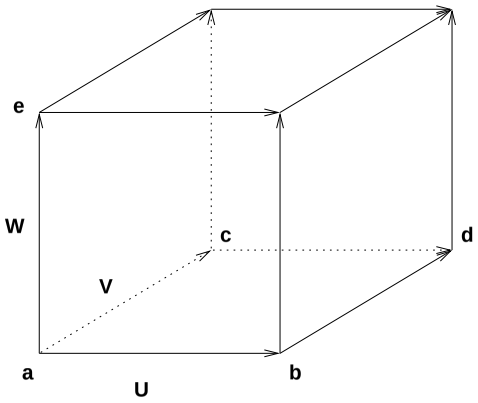


Figure 4: A 3-dimensional transition

Globular Modeling of HDA

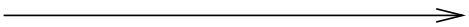


Figure 5: A 1-dimensional morphism

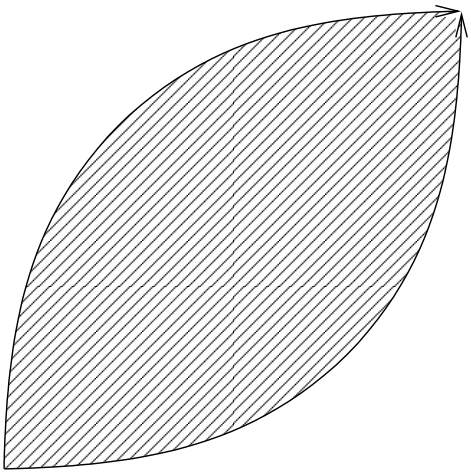


Figure 6: A 2-dimensional morphism

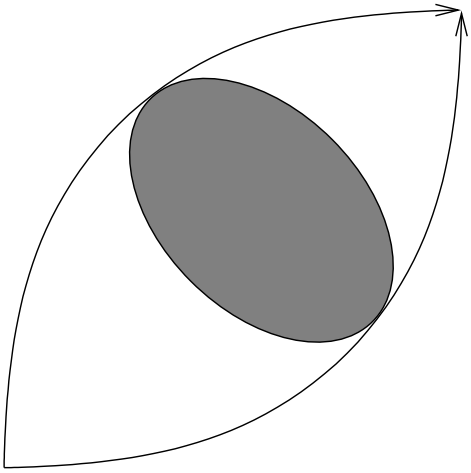


Figure 7: A 3-dimensional morphism

Globular ω-Category

A set A and for $n \geqslant 0$:

$$d_n^- = s_n : A \to A$$

$$d_n^+ = t_n : A \to A$$

$$*_n : \{(a,b) \in A \times A, t_n(a) = s_n(b)\} \to A$$

for $\alpha, \beta \in \{-,+\}$:

1. $d_m^\beta d_n^\alpha x = \begin{cases} d_m^\beta x \text{ if } m < n \\ d_n^\alpha x \text{ if } m \geqslant n \end{cases}$
2. $s_n x *_n x = x *_n t_n x = x$
3. $s_n(x *_n y) = s_n x$, $t_n(x *_n y) = t_n y$
4. for $m \neq n$, $d_m^\alpha(x *_n y) = d_m^\alpha x *_n d_m^\alpha y$
5. $(x *_n y) *_n z = x *_n (y *_n z)$
6. for $m \neq n$,
   $(x *_n y) *_m (z *_n w) = (x *_m z) *_n (y *_m w)$
7. $dim(x) = min\{n \geq 0, s_n x = t_n x = x\} < \infty$

Composing Higher Dimensional Morphisms

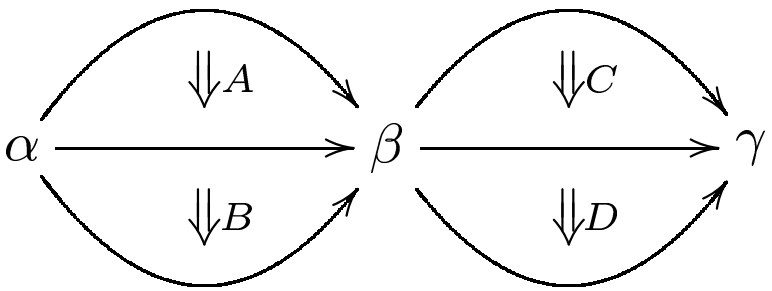


Figure 8: Composition of two 2-morphims and illustration of the globular and interchange law axioms

$$\alpha = s_0 s_1 A = s_0 t_1 A = s_0 s_1 B = s_0 t_1 B$$

$$\beta = t_0 s_1 A = t_0 t_1 A = t_0 s_1 B = t_0 t_1 B$$

$$t_1 A = s_1 B,\ t_1 C = s_1 D$$

$$(A *_1 B) *_0 (C *_1 D) = (A *_0 C) *_1 (B *_0 D)$$

$$t_1(A *_0 C) = t_1 A *_0 t_1 C = s_1 B *_0 s_1 D = s_1(B *_0 D)$$

Additional Axioms for *Kan Non-Contracting ω-Categories*

1.
- $(\mathbb{P}\mathcal{C})_n := \mathcal{C}_{n+1}$
- $s_n^{\mathbb{P}\mathcal{C}} := s_{n+1}^{\mathcal{C}}$
- $t_n^{\mathbb{P}\mathcal{C}} := t_{n+1}^{\mathcal{C}}$
- $*_n^{\mathbb{P}\mathcal{C}} := *_{n+1}^{\mathcal{C}}$

for $n \geqslant 0$ defines an ω-category called the *path ω-category* of $\mathcal{C}$

2. For any p-morphism x with $p \geqslant 2$ and any $n \geq 1$, there exists a p-morphism x' such that

$$x *_n x' = s_n x = t_n x'$$

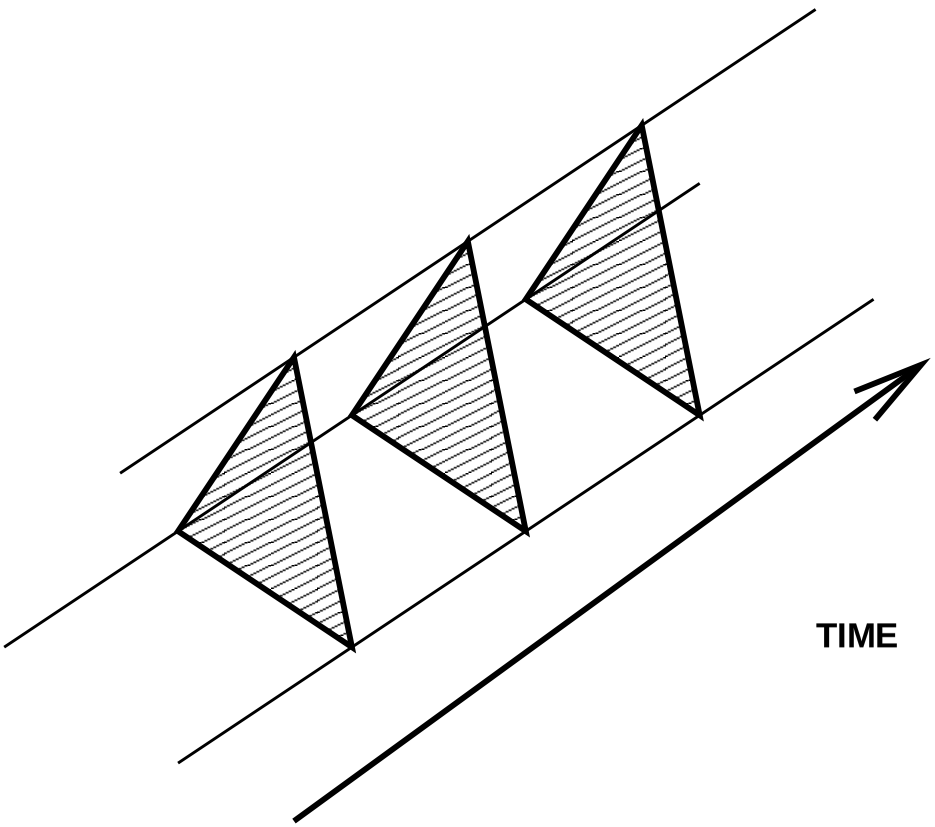
$$x' *_n x = s_n x' = t_n x$$

i.e. $\mathbb{P}\mathcal{C}$ ω-groupoid.

Section 2 : Algebraic Structure of Achronal Simplexes

Philosophy

1. **Detecting "achronal cuts" included in higher dimensional automata.**
2. **Getting information about the way that achronal cuts can slide along execution paths.**
3. **"Simple" geometric shapes must give rise to "finite" algebraic structures**
4.



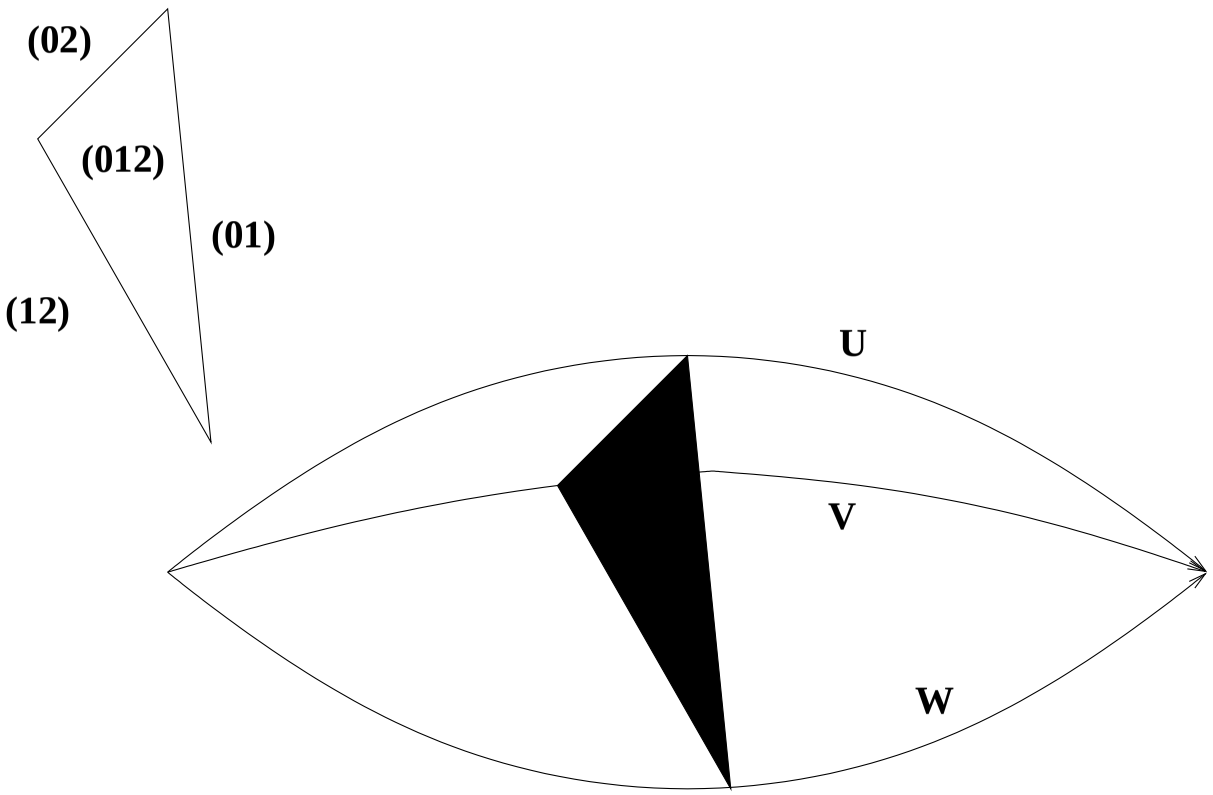


Figure 9: Globular 2-simplex

Figure 10: Negative and positive globular 2-simplexes

The Globular ω-Category Δ^n

1. **The underlying set of Δ^n is included in the set of subsets of non-empty strictly increasing sequences of elements of $\{0, 1, \ldots, n\}$**

2. **The underlying set of Δ^n contains all subsets like $R(\sigma)$ where $R(\sigma)$ subset of faces obtained from σ by removing elements of σ. E.g.**
   $R(013) = \{(013), (13), (03), (01), (0), (1), (3)\}$.

3. **All elements of Δ^n are compositions of $R(\sigma)$ where σ runs over O^n**

4. **For σ p-dimensional with $p \geqslant 1$**

$$s_{p-1}(R(\sigma)) = R\left(\{\partial_j \sigma, j \text{ is even}\}\right)$$
$$t_{p-1}(R(\sigma)) = R\left(\{\partial_j \sigma, j \text{ is odd}\}\right)$$

   where $\sigma = \{\sigma_0 < \ldots < \sigma_p\}$ implies
   $\partial_j \sigma := \{\sigma_0 < \ldots < \widehat{\sigma_j} < \ldots < \sigma_k\}$.

5. **if $\Sigma, T \in \Delta^n$ and $t_p(\Sigma) = s_p(T)$ for some p, then $\Sigma \cup T \in \Delta^n$ and $\Sigma \cup T = \Sigma *_p T$.**

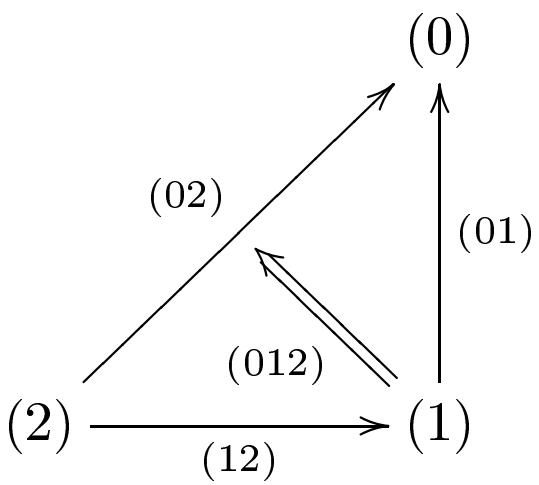


Figure 11: The ω-category Δ^2

The Equivalence Relation $\mathcal{R}^\alpha$

Let $\mathcal{C}$ be a non-contracting ω-category. Denote by $\mathcal{R}^-$ (resp. $\mathcal{R}^+$) the reflexive symmetric and transitive closure of $\{(x, x *_0 y), x, y, x *_0 y \in \mathbb{P}\mathcal{C}\}$ (resp. $\{(y, x *_0 y), x, y, x *_0 y \in \mathbb{P}\mathcal{C}\}$) in $\mathbb{P}\mathcal{C} \times \mathbb{P}\mathcal{C}$.

Semi-Path ω-Category $\mathbb{P}^\alpha\mathcal{C}$ of $\mathcal{C}$

Let $\alpha \in \{-, +\}$ and let $\mathcal{C}$ be a non-contracting ω-category. Then the universal problem

> **"There exists a pair $(\mathcal{D}, \mu)$ such that $\mathcal{D}$ is an ω-category and μ an ω-functor from $\mathbb{P}\mathcal{C}$ to $\mathcal{D}$ such that for any $x, y \in \mathbb{P}\mathcal{C}$, $x\mathcal{R}^\alpha y$ implies $\mu(x) = \mu(y)$."**

has a solution $(\mathbb{P}^\alpha\mathcal{C}, (-)^\alpha)$.

The ω-category $\mathbb{P}^-\mathcal{C}$ (resp. $\mathbb{P}^+\mathcal{C}$) is called the negative (resp. positive) semi-path ω-category of $\mathcal{C}$.

Intuitively, $\mathbb{P}^-\mathcal{C} = \mathbb{P}\mathcal{C}/\{x = x *_0 y, dim(x) \geq 1\}$ and $\mathbb{P}^+\mathcal{C} = \mathbb{P}\mathcal{C}/\{y = x *_0 y, dim(x) \geq 1\}$

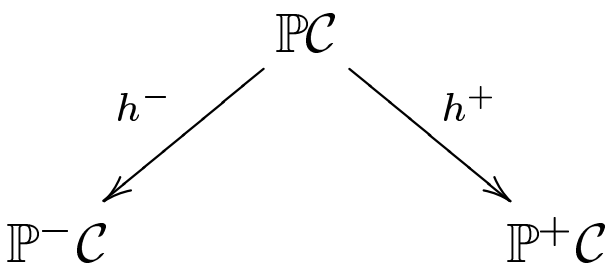


Figure 12: The fundamental diagram

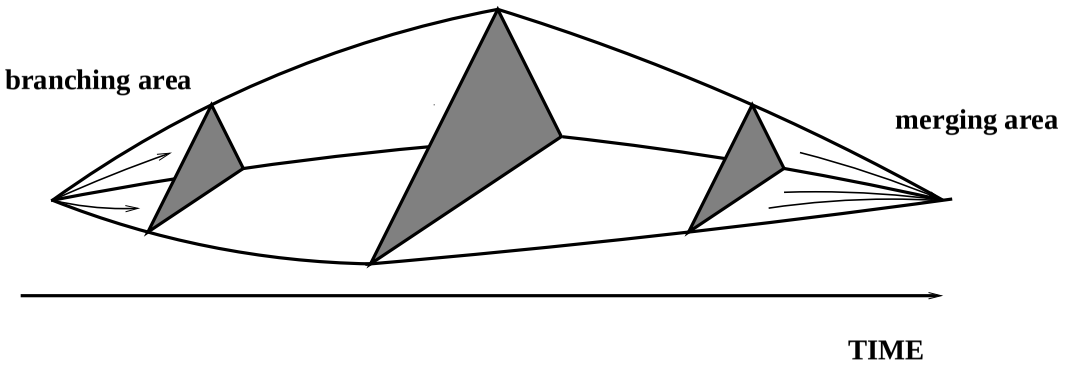


Figure 13:

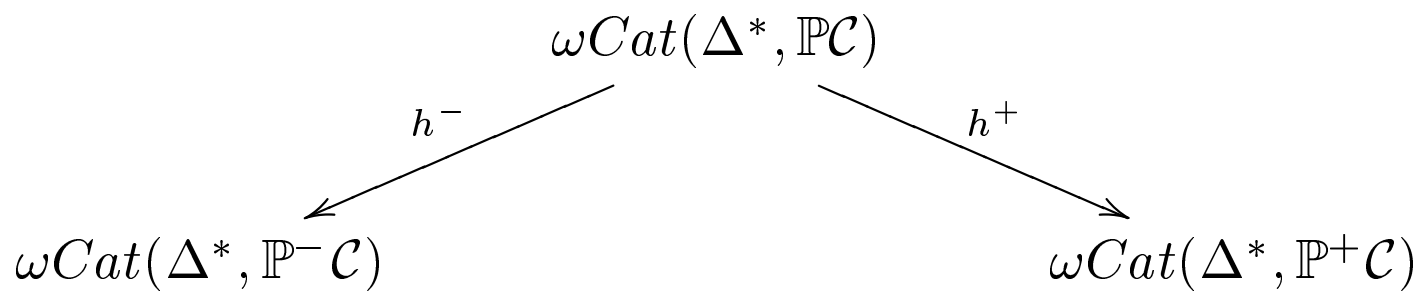


Figure 14: The augmented simplicial version of the fundamental diagram

with the augmentation maps :

1. $\partial_{-1}(x) = (s_0 x, t_0, x)$ **for** $x \in \omega Cat(\Delta^0, \mathbb{P}\mathcal{C})$
2. $\partial_{-1}(x) = s_0 x$ **for** $x \in \omega Cat(\Delta^0, \mathbb{P}^-\mathcal{C})$
3. $\partial_{-1}(x) = t_0 x$ **for** $x \in \omega Cat(\Delta^0, \mathbb{P}^+\mathcal{C})$

Geometric Content of The Homotopy and Homology Groups

1. $H^{gl}_{n+1}(\mathcal{C}) := H_n(\mathcal{N}^{gl}(\mathcal{C}))$ **:** $(n+1)$**-dimensional oriented globes** $(n \geq 0)$**.**

2. $H^{gl^-}_{n+1}(\mathcal{C}) := H_n(\mathcal{N}^{gl^-}(\mathcal{C}))$ **:** $(n+1)$**-dimensional branching areas** $(n \geq 0)$**.**

3. $H^{gl^+}_{n+1}(\mathcal{C}) := H_n(\mathcal{N}^{gl^+}(\mathcal{C}))$ **:** $(n+1)$**-dimensional merging areas** $(n \geq 0)$**.**

4. $\pi^{gl}_{n+1}(\mathcal{C}, \phi) := \pi_n((\mathcal{N}^{gl}(\mathcal{C})), \phi)$ **: oriented** $(n+1)$**-dimensional cylinders with achronal basis** $(n \geq 1)$**.**

5. $\pi^{gl^-}_{n+1}(\mathcal{C}, \phi) := \pi_n((\mathcal{N}^{gl^-}(\mathcal{C})), \phi)$ **: "germ" of beginning of** $(n+1)$**-dimensional oriented cylinders with achronal basis** $(n \geq 1)$**.**

6. $\pi^{gl^+}_{n+1}(\mathcal{C}, \phi) := \pi_n((\mathcal{N}^{gl^+}(\mathcal{C})), \phi)$ **: "germ" of ending of** $(n+1)$**-dimensional oriented cylinders with achronal basis** $(n \geq 1)$**.**

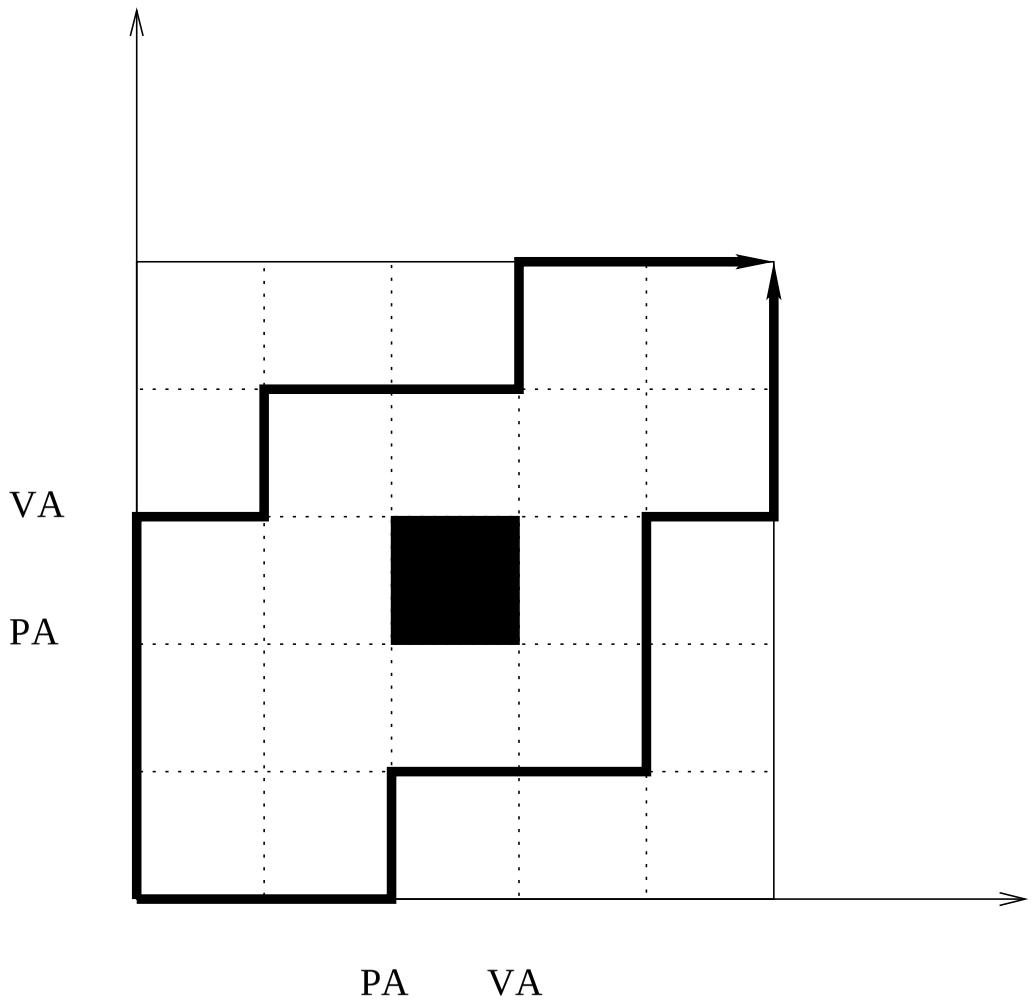


Figure 15: Example from distributed database theory : two 1-morphisms are depicted

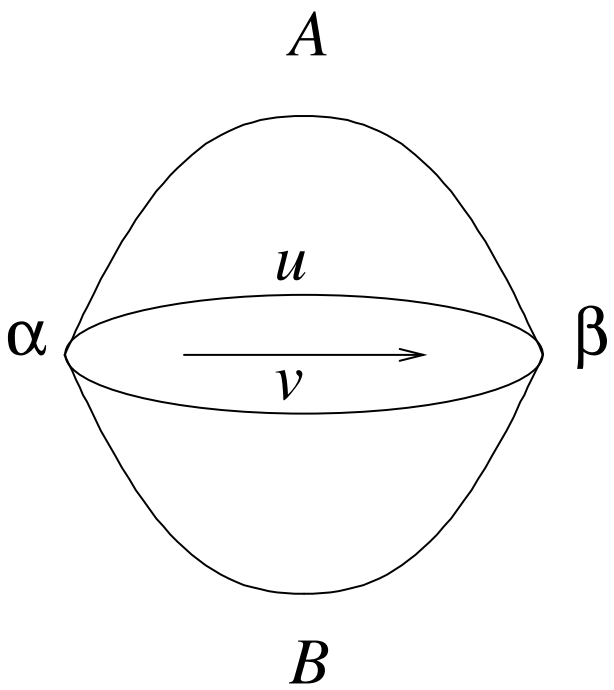


Figure 16: A 3-dimensional hole

Section 3 : Deformation of Higher Dimensional Automata

Deformations Leaving Invariant The Structure of The Flow of Execution Paths :

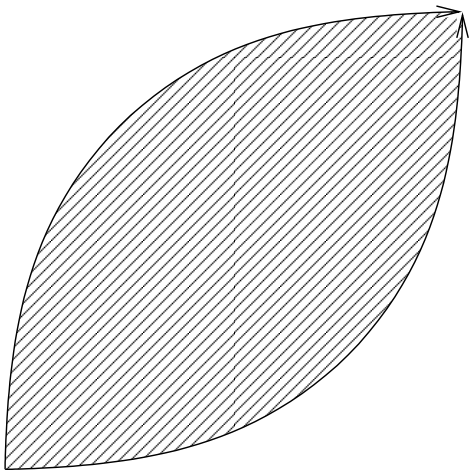
1. **The *spatial deformations* (i.e. deformation of achronal cuts up to homotopy) : contraction or dilatation of p-morphisms for $p \geq 2$ (Figure 17)**
2. **The *temporal deformations* (closely related to the notion of homeomorphism) : adding states in the middle of 1-morphisms (Figure 18)**

Fundamental rule :

In any case an oriented segment must be contracted in the same equivalence class !

Spatial Deformations

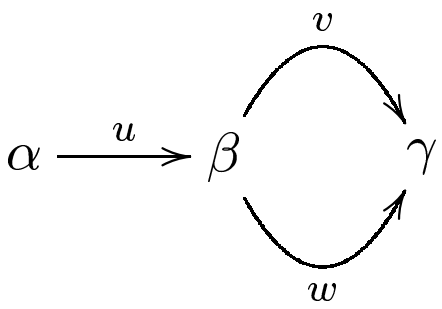
(a) The 1-dimensional globe



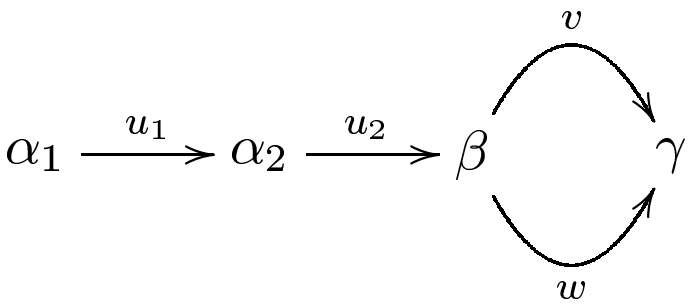
(b) The 2-dimensional globe

Figure 17: Example of spatial deformation

Temporal Deformations



(a) A 1-dimensional HDA



(b) Subdivision of u in X

Figure 18: Example of temporal deformation

The Swiss Flag Example

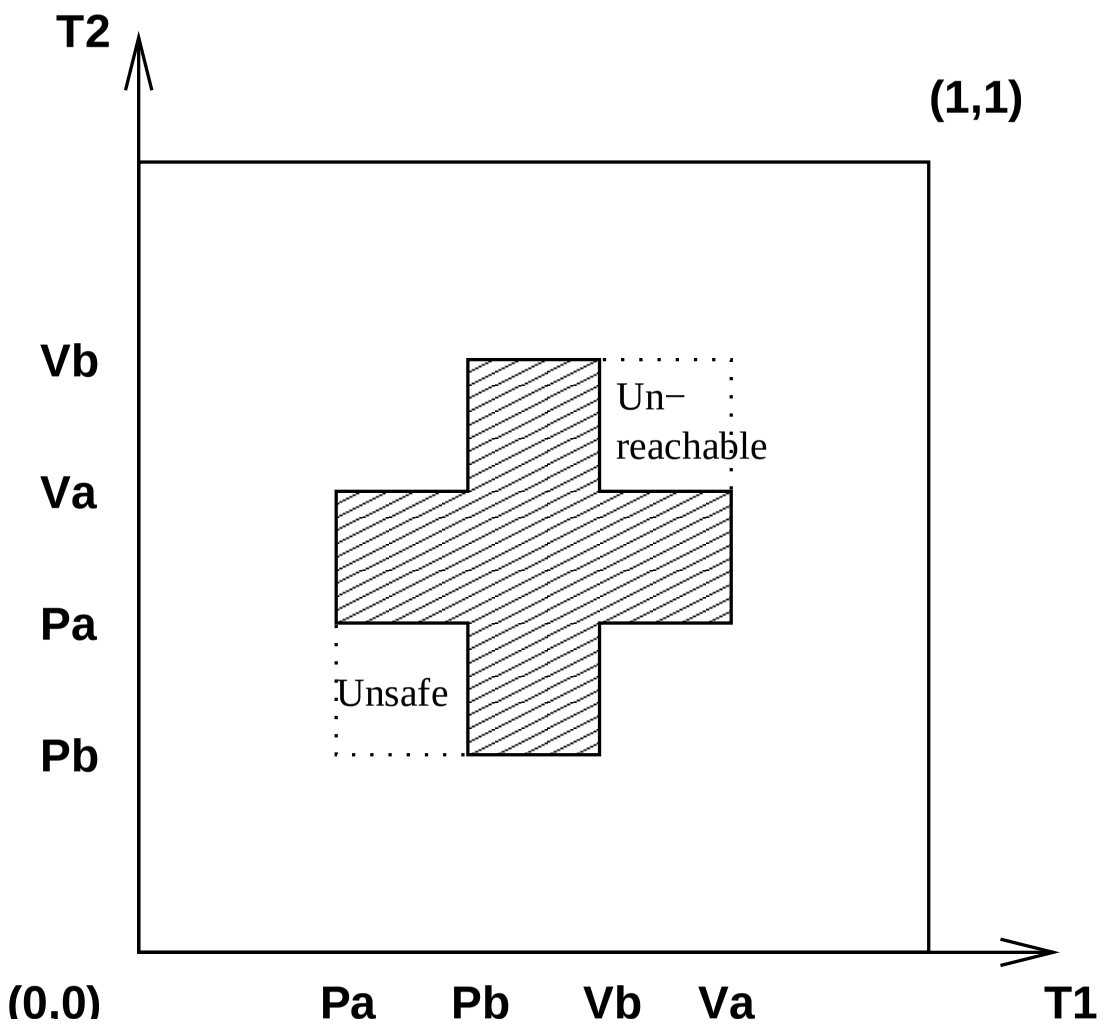


Figure 19: Example of a progress graph

Functors	**S-invariant**	**T-invariant**	**Kan**	$-\infty \to +\infty$
$\mathcal{N}^{gl}$	**yes**	**no**	**yes**	**yes**
$\mathcal{N}^{\pm}$	**almost**	**no**	**almost never**	**yes**
H^{gl}	**yes**	**no**	**no meaning**	**yes**
$H^{\pm}$	**yes**	**yes**	**no meaning**	**yes**
$\mathcal{N}^{gl^{\pm}}$	**yes**	**no**	**yes**	**yes**
$H^{gl^{\pm}}$	**yes**	**yes**	**no meaning**	**yes**
π^{gl}	**yes**	**yes**	**no meaning**	**no**
$\pi^{gl^{\pm}}$	**yes**	**yes**	**no meaning**	**no**
$\mathcal{N}^{bigl}$	**yes**	**yes**	**yes ?**	**yes**
H^{bigl}	**yes**	**yes**	**no meaning**	**yes**
(π^{bigl})	**(yes)**	**(yes)**	**(no meaning)**	**(yes)**

Figure 20: Behavior w.r.t the two types of deformations

Explanation of Figure 20

1. **S-invariance of $\mathcal{N}^{gl}$, $\mathcal{N}^{gl^{\pm}}$: a S-deformation corresponds to a usual deformation up to homotopy.**
2. **$\mathcal{N}^{bigl}$ is a composite of a T-invariant functor (the classifying space functor) and of a S-invariance functor.**
3. **H^{gl}, $H^{gl^{\pm}}$, π^{gl}, $\pi^{gl^{\pm}}$ and H^{bigl} inherit the S-invariance of the above nerves.**
4. **π^{gl} and $\pi^{gl^{\pm}}$ need a base-point (of $\mathbb{P}\mathcal{C}$), i.e an execution path of $\mathcal{C}$ since $\mathcal{C}_1 = (\mathbb{P}\mathcal{C})_0$. So π^{gl} and $\pi^{gl^{\pm}}$ contain information only about achronal cuts crossing the "base-point".**

T-Invariance of $H^{gl^{\pm}}$

1. **A *thin element* of $\mathcal{N}^{gl^{\pm}}(\mathcal{C})$ is an element of zero volume. $x \in \mathcal{N}_n^{gl^{\pm}}(\mathcal{C})$ thin iff the dimension of the interior of x is of dimension at most n.**
2. **For $\mathcal{C}$ freely generated by a precubical set, these elements do not generate (conjecturally) non-trivial homology classes.**
3. **Modulo thin elements**
   - $x(01\dots n) *_p y(01\dots n) = z(01\dots n)$ **for** $p \geq 1$ **implies** $x + y = z$
   - $x(01\dots n) *_0 y(01\dots n) = z(01\dots n)$ **implies** $x = z$**.**

Section 4 : The Directed Hurewicz Morphism

The Biglobular Nerve

Let $\mathcal{C}$ be a non-contracting ω-category.

For $x, y : \Delta^n \to \mathbb{P}\mathcal{C}$,

1. $\{S(x)\} = s_0 \circ x(\Delta^n)$
2. $\{T(x)\} = t_0 \circ x(\Delta^n)$
3. **For $(\sigma_0 \dots \sigma_r) \in \Delta^n$,**

$$(x{*}y)((\sigma_0 \dots \sigma_r)) := x((\sigma_0 \dots \sigma_r)){*_0} y((\sigma_0 \dots \sigma_r)).$$

Then S, T and $*$ yield a structure of small category on $\mathcal{N}_n^{gl}(\mathcal{C}) \cup \{$constant maps $\Delta^n \to \mathcal{C}_0\}$.

One obtains an augmented simplicial object in the category of small categories since $\partial_i(x * y) = \partial_i(x) * \partial_i(y)$ and $\epsilon_i(x * y) = \epsilon_i(x) * \epsilon_i(y)$ for $0 \leq i \leq n$.

By composing by the classifying space functor of small categories, one obtains a bisimplicial set which is called the *biglobular nerve*.

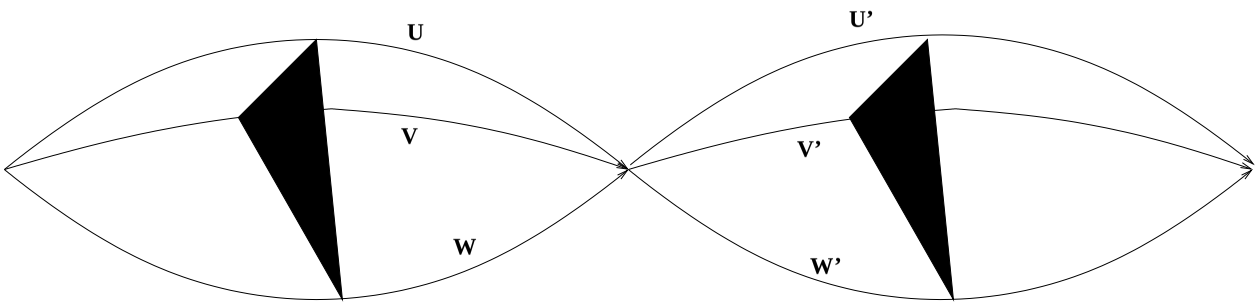


Figure 21: Composition of two 2-dimensional globular simplexes

Constructing The Directed Hurewicz Morphism

Constructing a functor π_n^{bigl} for $n \geq 1$ such that

1. **π_n^{bigl} is S-invariant and T-invariant.**
2. **π_n^{bigl} contains in some way all π_n homotopy groups of all achronal cuts.**
3. **There exists a natural transformation $\pi_n^{bigl} \to H_n^{bigl}$ containing in some way all usual Hurewicz morphisms for all achronal cuts.**
4. **π_n^{bigl} contains information concerning all achronal cuts from $t = -\infty$ to $t = +\infty$.**