Topology, geometry and computational complexity

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with

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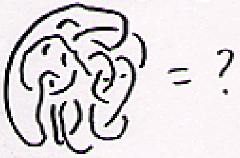
General goals and questions:

1. How hard are the problems of 3-dimensional topology?

eg Classifying knots and 3-manifolds.



b. Can the technology of 3-dimensional topology say something about the relationships between various complexity classes?

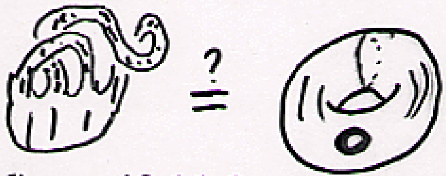
(as with Jaco-Shalen-Johannson decomposition for finitely generated groups).

c. The complexity of numerous problems has been analyzed. Do the techniques developed in studying complexity have something to say about 3-dimensional topology and geometry?

Yes.

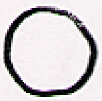
Some Other Questions: Find a procedure to determine

* When are two manifolds the same? (homeomorphic)



* When are
Can we recognize (closed) manifolds?

Dimension 1 & 2: Easy.



$\chi(F) = \boxed{e-v+f}$ CLASSIFIES

Dimension 3: Special cases can be done.

3-sphere, Haken, Lens spaces, ...
(Rubinstein-Thompson, Haken, Rubinstein)

Dimensions > 3 : Many interesting problems are undecidable.

Classifying 4-manifolds, Novikov, Boone, Markov
Recognizing S^5,...
Nabutovsky

Dimension 3 seems the most interesting.

* Problems are just barely solvable.

* Complexity seems to be on the edge between polynomial and exponential.

* Computer programs (eg SNAPPEA by Jeff Weeks and collaborators) are already a major tool for 3-manifold research.
How far can they go?

(2000 ⇝ About 30 crossing knots)

Analyzing complexity leads to insights into what to do, and how to do it better.

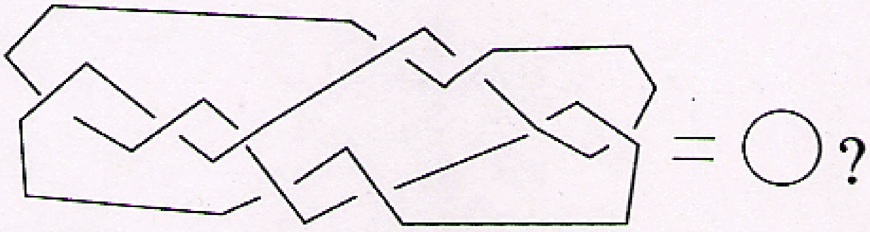
* Can the tools of 3-manifold theory shed new light on the relation between complexity classes?

Some Key 3-dimensional Problems:

1. UNKNOTTING PROBLEM

Instance: A link diagram D .

Question: Is D a knot diagram that represents the trivial knot?



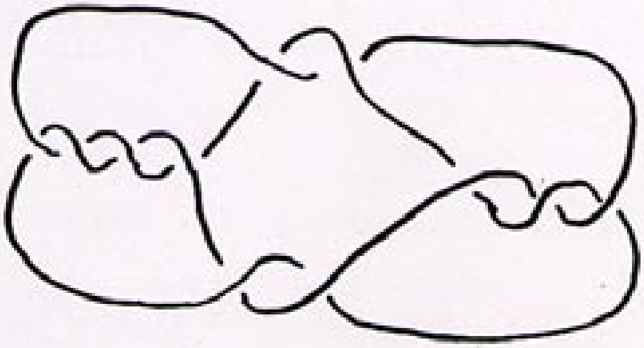
This is a special case of

2. KNOT EQUIVALENCE PROBLEM:

Instance: Two link diagrams D_1 and D_2

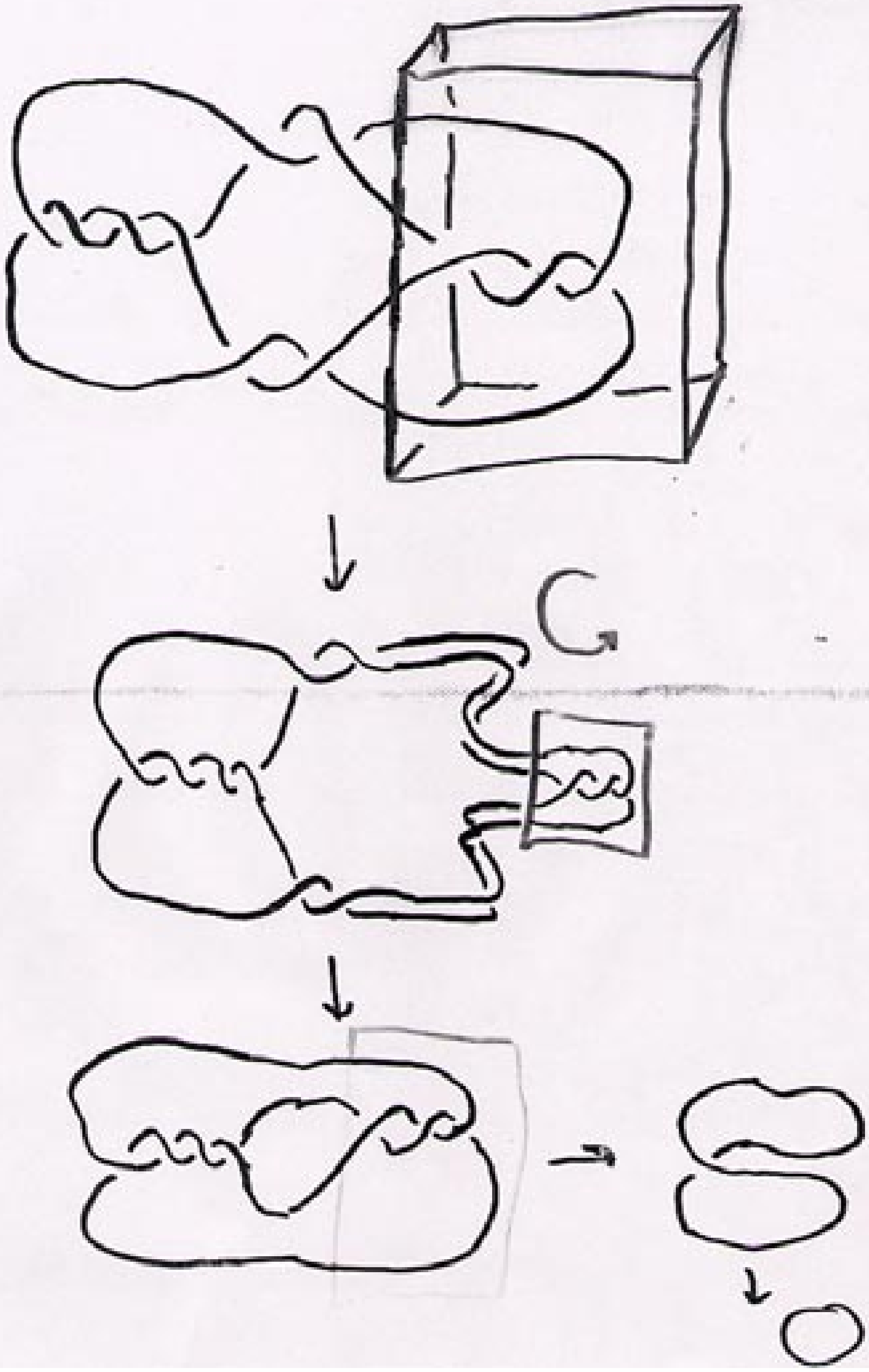
Question: Are D_1 and D_2 diagrams of equivalent knots?

NOT SO EASY EXAMPLE OF KNOT RECOGNITION

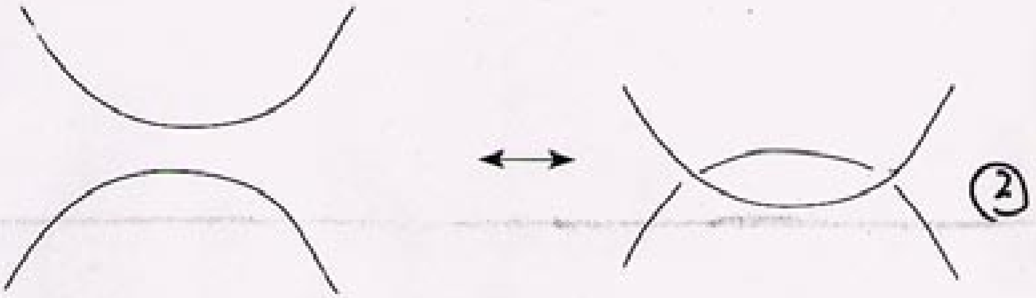


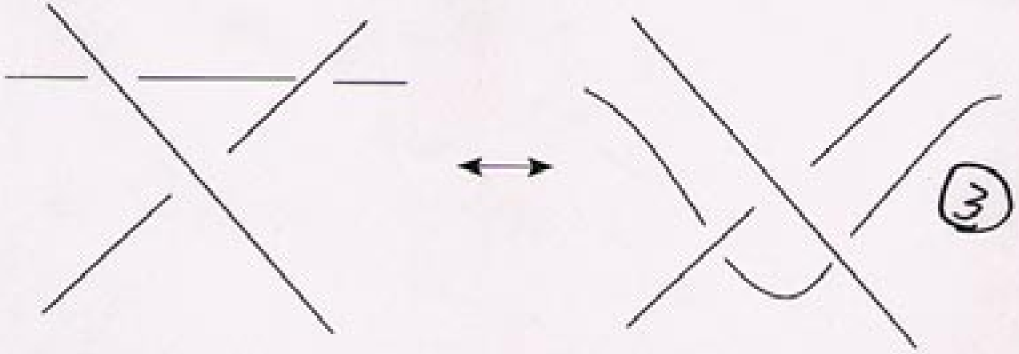
|| ?

Goeritz 1934



Reidemeister moves suffice to move between any two diagrams of a knot



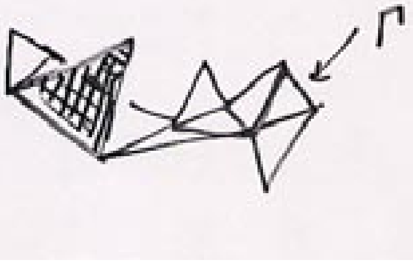


BUT HOW MANY ARE NEEDED?

THEOREM (H-LAGARIAS) (2000)

IF Γ IS UNKNOTTED WITH N CROSSINGS, THEN IT CAN BE DEFORMED TO ○ WITH AT MOST C^N REIDEMEISTER MOVES. ($C>1$ A constant)

IDEA: Find a triangulated disk spanning Γ.



Collapsing triangles
↓
Reidemeister Moves

A Key Tool for algorithms in 3-manifolds:

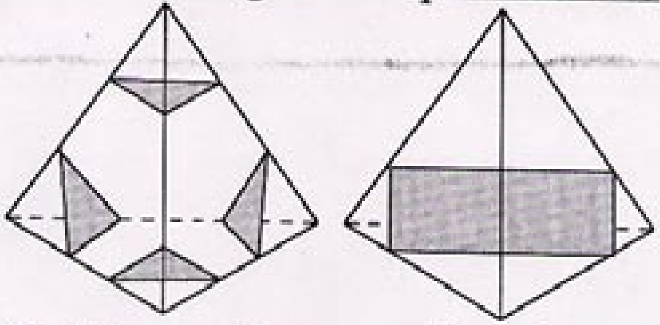
Normal Surfaces

(Kneser 1929, Haken 1961)

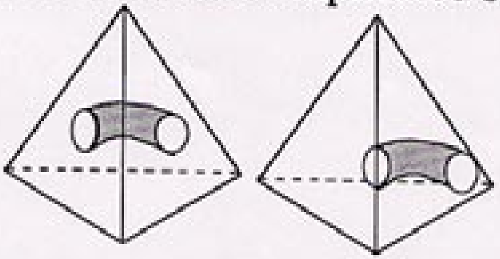
(+ JACO, OERTEL, TOLLEFSON, RUBINSTEIN
THOMPSON, CASSON)

Normal surfaces give an efficient way to understand surfaces in a triangulated three dimensional. They have several important properties:

a. Locally controlled. They intersect each tetrahedron in triangles and quadrilaterals only.

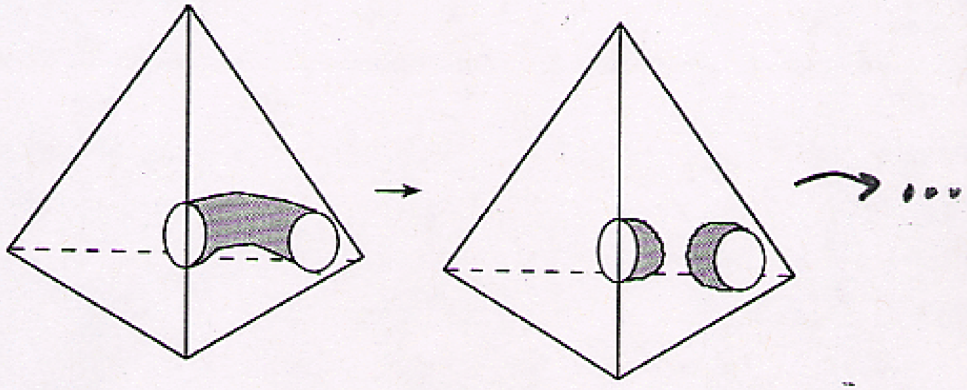


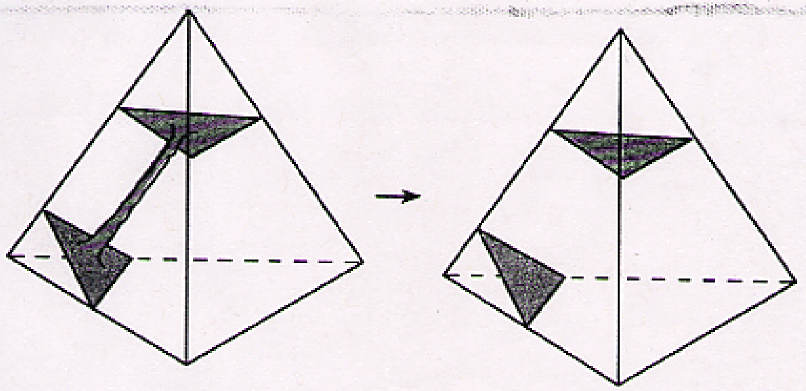
NOT in tubes or other complicated structures.



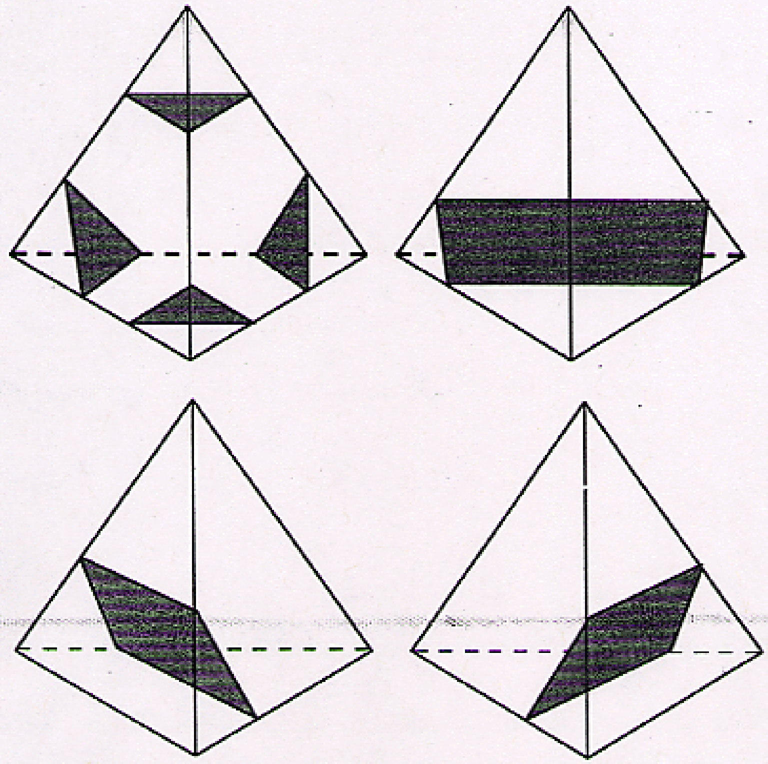
NORMALIZING

Surfaces can be pushed around in a 3-dimensional manifold until abnormal behavior is eliminated.





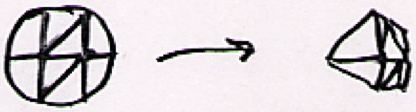
The resulting surface is normal.
(May be empty)



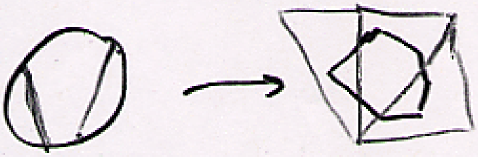
There are four kinds of triangle and three kinds of quadrilateral in each tetrahedron. A normal surface is determined by counting how many of each type there are in each tetrahedron.

ESSENTIAL PROPERTY OF NORMAL SURFACES:

THEIR PL-STRUCTURE (OR TRIANGULATION) REFLECTS THE GEOMETRY AND TOPOLOGY OF THE IMAGE SPACE, NOT THE DOMAIN.



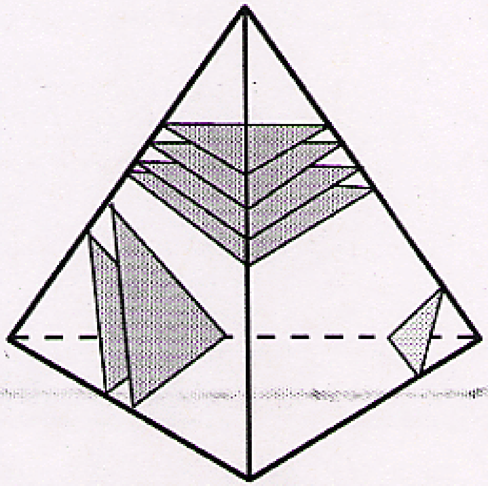
collapsing triangles



limited local types

A normal surface is determined by a vector of 7t non-negative integers

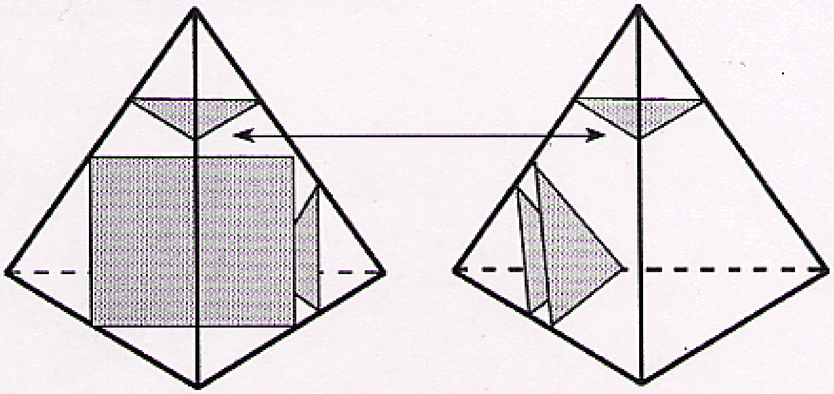
$(v_1, v_2, v_3, \ldots\ v_{7t})$



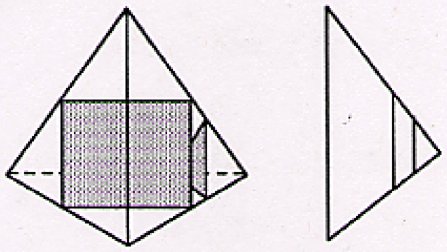
Of the seven types of triangle and quadrilateral in this tetrahedron, three appear. The vector corresponding to this normal surface looks like

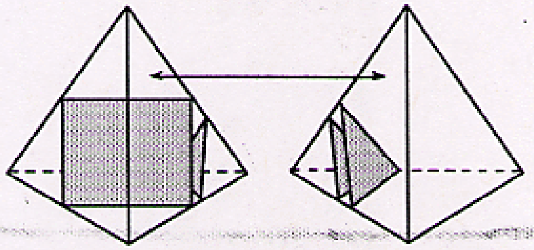
$(v_1, v_2, v_3, \ldots\ v_{7t}) = (\ldots, 0, 4, 2, 1, 0, 0, 0\ \ldots)$

2, They match up along the common faces of pairs of tetrahedra in M.



Not all vectors give normal surfaces. The pieces must match up across tetrahedra with common faces.





This leads to linear equations for the coordinates of the vector $(v_1, v_2, v_3, \ldots\ v_{7t})$, of the form

$$v_i + v_j = v_k + v_l$$

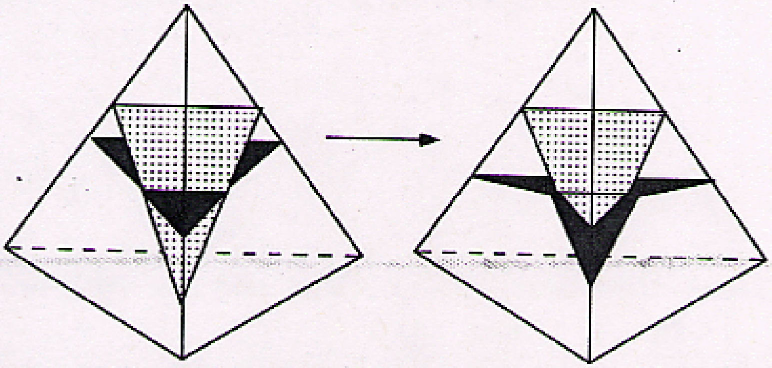
v_i here counts the number of one type triangle in a tetrahedron, while v_j counts one type of quadrilateral.

Also have : $v_i \geq 0$, $(V_i V_j = 0)$

Finding normal surfaces can now be formulated algebraically.
A problem in integer linear programming.

Normal surfaces -- integer vectors
$(v_1, v_2, v_3, \ldots\ v_{7t})$
satisfying linear eqns

sum of surfaces -- sum of vectors



Exploiting this gives:
Any normal surface can be written as a sum of a finite collection of fundamental surfaces.

A surface C is fundamental if it cannot be written as as non-trivial sum.

$$C \neq A + B$$

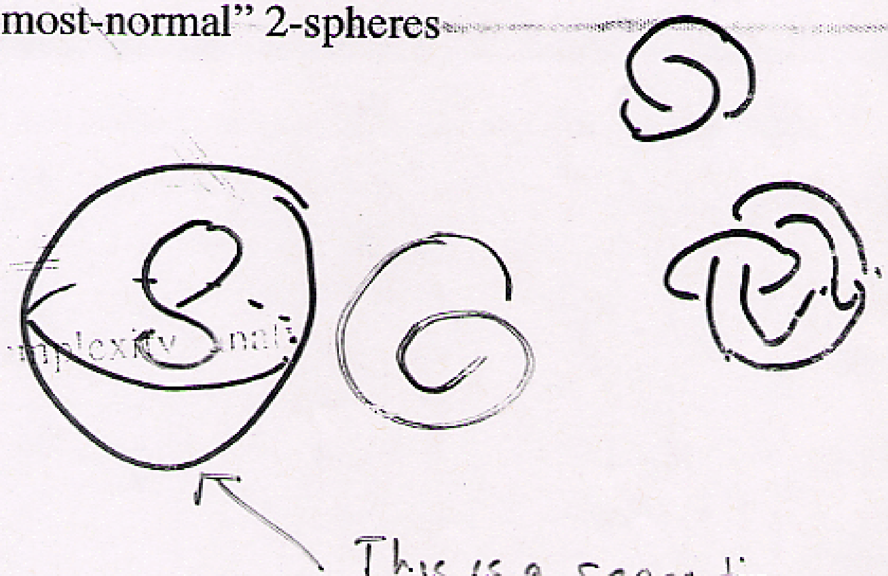
Fundamental surfaces have bounded size, which can be computed.

In many cases one can show that interesting classes of surfaces have representatives among the finite collection of fundamental surfaces.

eg
Unknotting disks

Separating 2-spheres

"Almost-normal" 2-spheres



This is a separating 2-sphere for the Brown link

Analyzing Haken's Unknotting Algorithm

K is unknotted <=> K is the boundary of an embedded disk

Triangulate and normalize:

<=> K is the boundary of a normal disk

Analyze how surfaces sum:

<=> K is the boundary of a fundamental normal disk

There are finitely many fundamental normal surfaces, and they can be checked one by one to see if any is a disk with boundary K.

APPLICATIONS OF NORMAL CURVES AND NORMAL SURFACES:

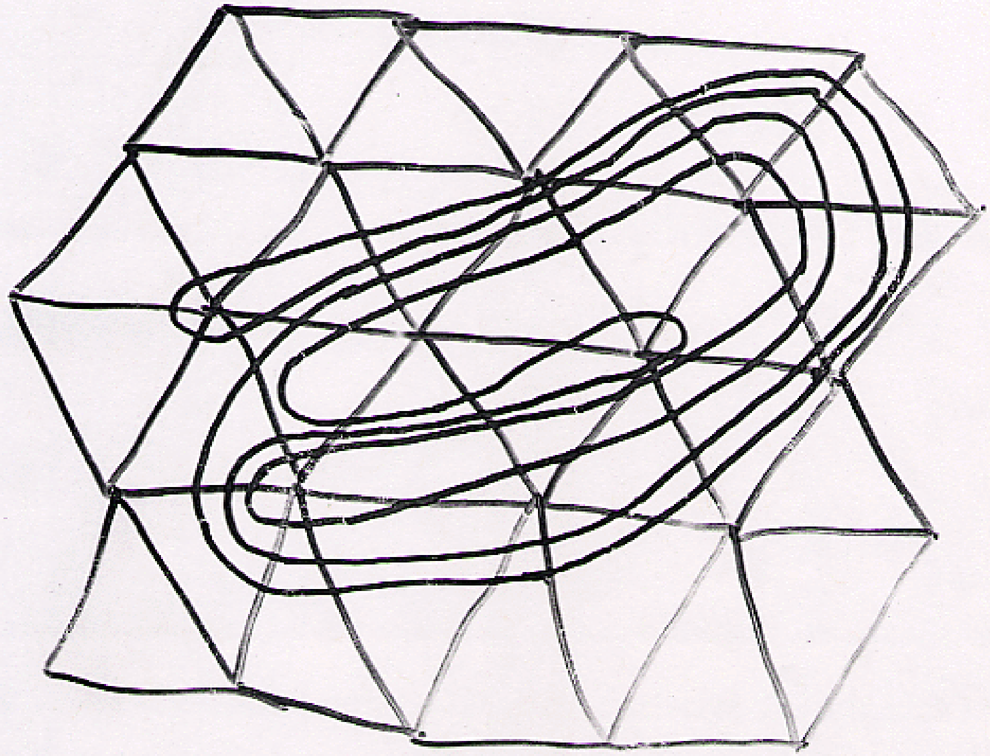
* Designed to solve algorithmic problems in 3-manifolds

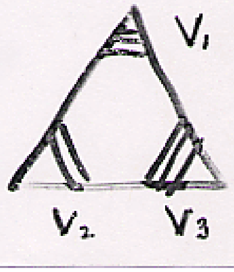
<u>Question</u> Do they have applications in

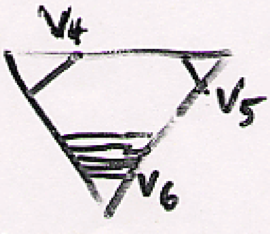
a) surface and curve representation (in computer graphics)

b) surface and curve reconstruction

c) scientific computation (front evolution) ?

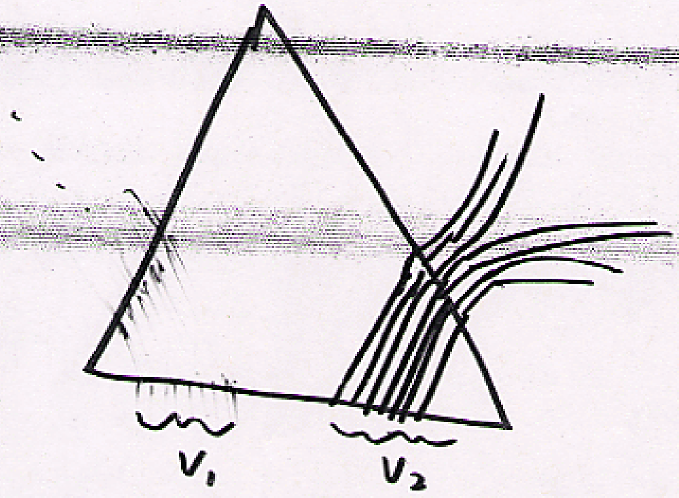






$V_1 + V_3 = V_4 + V_6$

In certain settings they are exponentially more efficient than commonly used surface representations.



NORMAL REPRESENTATION REQUIRES $\log(V_1) + \log(V_2)$

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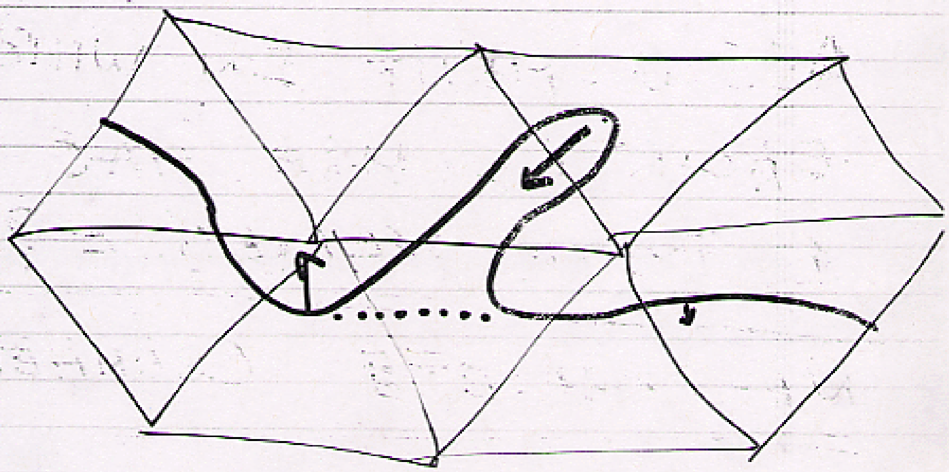
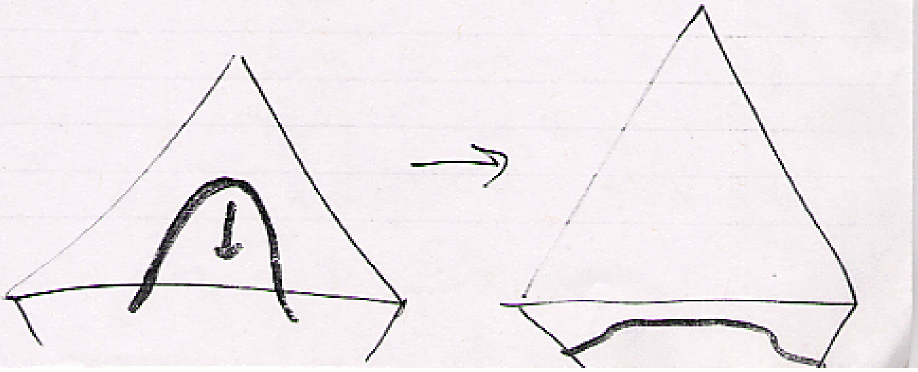
Scientific Computation

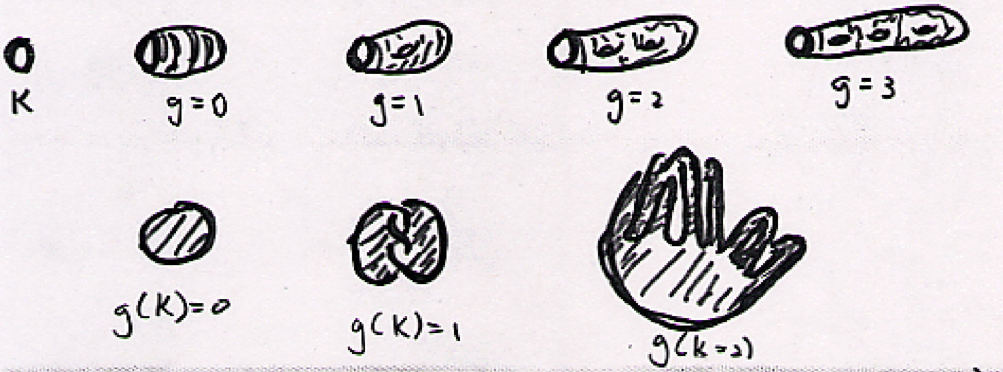
Curve and surface evolution
(SHIFT & ITERATE)





A RECENT RESULT

Problem: 3-MANIFOLD KNOT GENUS
INSTANCE: A 3-dimensional simplical complex R, a 1-dimensional subcomplex K, and a natural number g.
QUESTION: Does the 1-complex represent a knot K with genus(K) $\leq$ g?



$$\text{genus}(K) = \min(\text{genus}(S) : K = \partial S)$$

It was previously known that 3-MANIFOLD KNOT GENUS is in PSPACE. (H-Lagarias-Pippenger, 1997)

Theorem (Agol-H-Thurston)
3-MANIFOLD KNOT GENUS is **NP-complete**.

$\exists$ OTHER CONNECTIONS OF MINIMAL SURFACES TO ALGORITHMS AND COMPLEXITY.

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e.g.

Theorem (AGOL-H-Thurston)

Γ is a curve in the 1-skeleton of a Piecewise Flat 3-Manifold, with t tetrahedra. Deciding if Γ bounds a surface with Area $< k$, $k \in \mathbb{Z}$ a given integer, is NP-HARD.

IDEA: THERE IS A PROBLEM CALLED **SAT** WHICH IS NP-HARD

INSTANCE: A Boolean Formula

$$(X_1 \vee X_3 \vee \overline{X_6}) \wedge (X_2 \vee \overline{X_3} \vee X_1)$$

$$\wedge \ldots \underbrace{(X_{17} \vee \overline{X_1} \vee X_2)}_{\text{CLAUSES}}$$

QUESTION:

IS THERE A TRUTH ASSIGNMENT TO $X_1, X_2, .. X_N$ which makes the expression TRUE?

TRANSFORM THIS TO A QUESTION ABOUT WHETHER A CURVE SPANS A SMALL AREA F

VARIATION

1-IN-3 SAT:

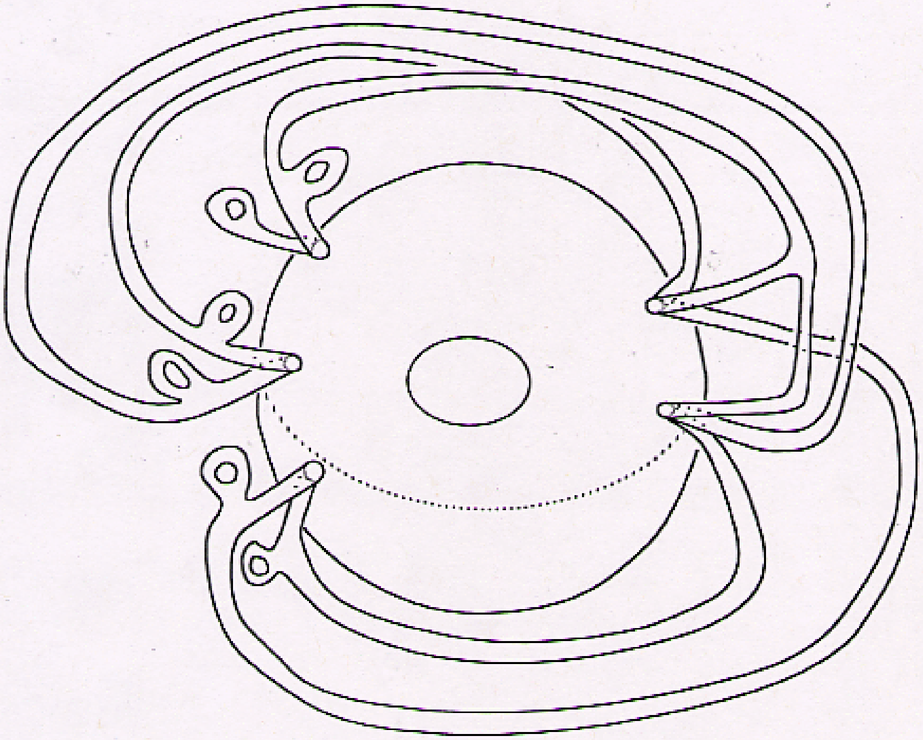
$(X_1 \vee X_2 \vee X_3) \wedge (X_1 \vee \overline{X}_2 \vee \overline{X}_3)$

IS THERE A TRUTH ASSIGNMENT IN WHICH EACH CLAUSE HAS 1 TRUE VARIABLE?

NP-COMPLETE (SCHAEFER)

HERE: YES. Set

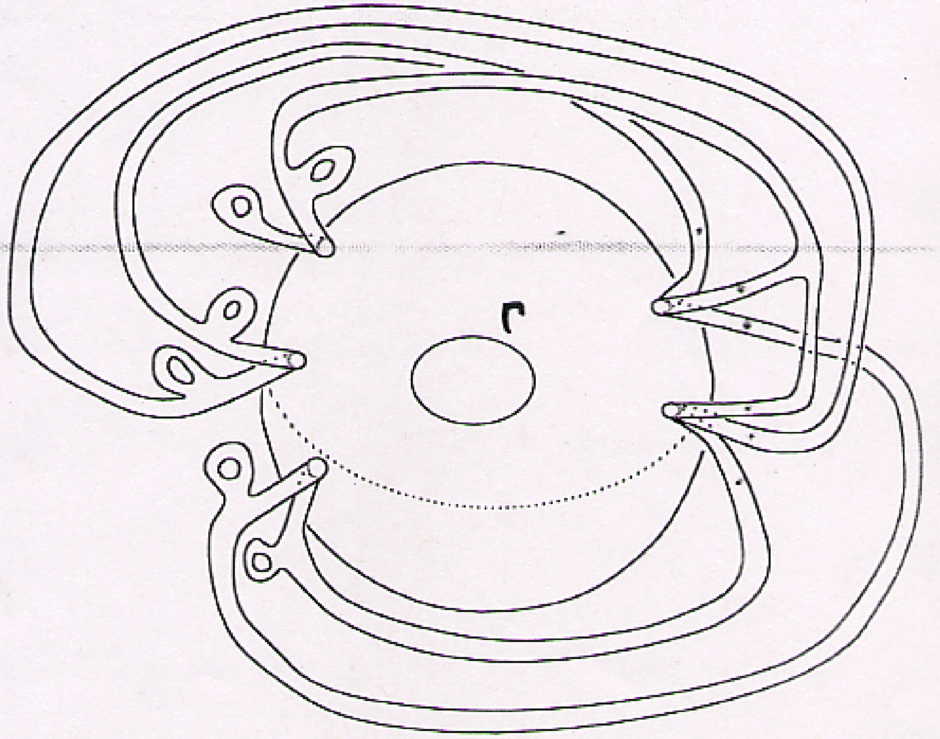
$\overline{X}_1, X_2, \overline{X}_3$ true.



Does Γ bound a surface of genus ≤ 5 ?

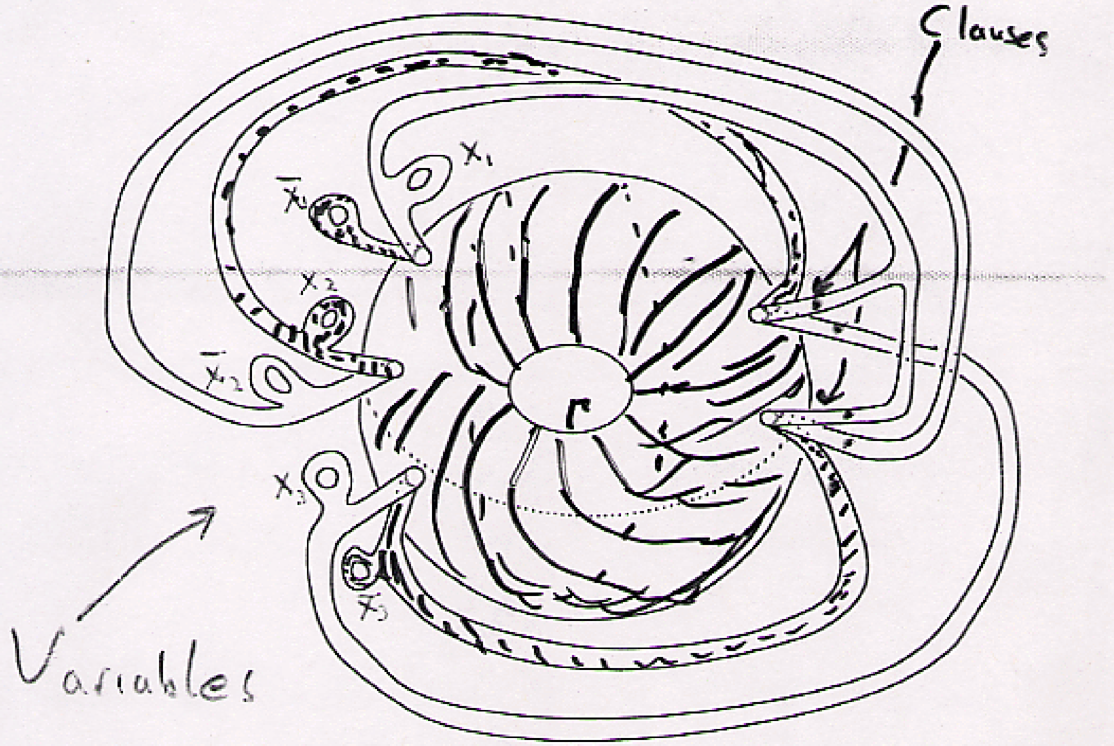
or

Does Γ bound a surface of area ≤ 2.5 ?



$(X_1 \vee X_2 \vee X_3) \wedge (x_1 \vee \bar{x}_2 \vee \bar{x}_3)$

IS THERE A TRUTH ASSIGN. WITH ONE OF EACH CLAUSE BEING A SATISFIED VARIABLE?

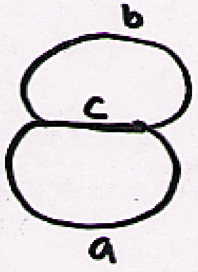


YES! $\bar{x}_1$, x_2, $\bar{x}_3$

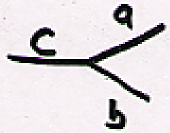
ORBIT COUNTING ALGORITH

SIMPLE EXAMPLE:

WEIGHTED GRAPHS

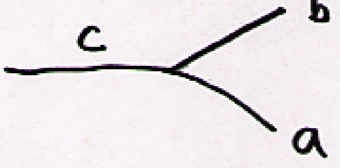


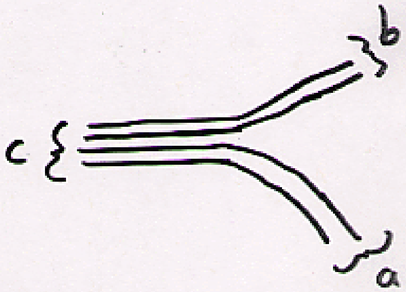
$a, b, c \in \mathbb{Z}_+$



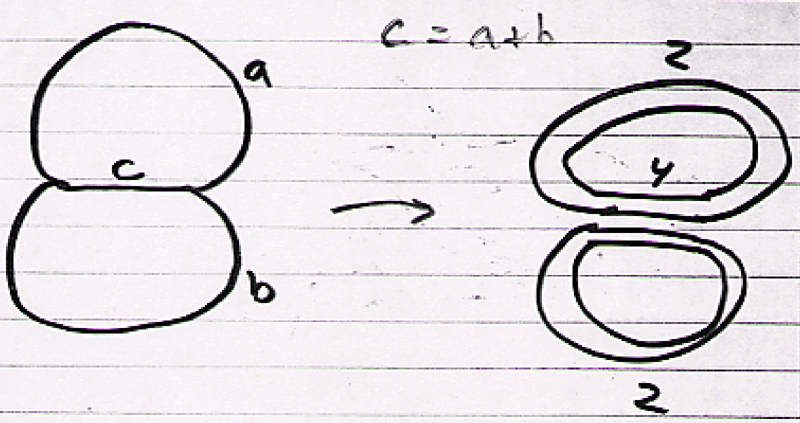
$\Rightarrow c = a + b$

Can associate a collection of curves to this structure

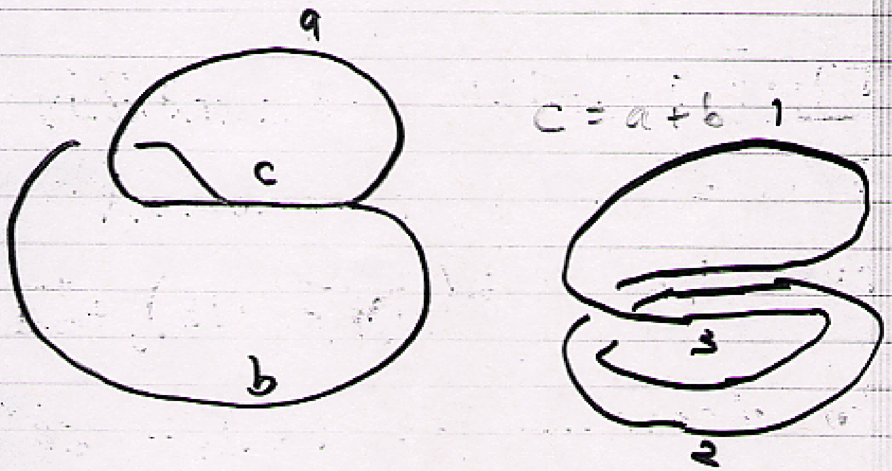




Q: HOW MANY COMPONENTS?

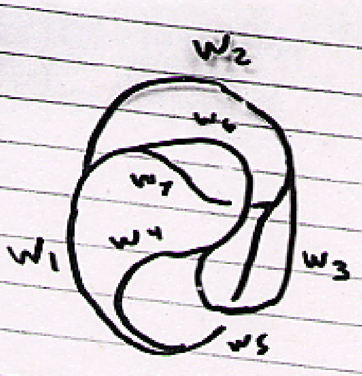


C components



NUMBER OF COMPONENTS

$= ?\ \gcd(a,b)$



Number of components = ?

THEOREM (AGOL - H - THURSTON)

THERE IS AN ALGORITHM, POLYNOMIAL IN $\log(\Sigma w_i)$ ~~and~~ which counts the number of components of the associated curve.

COROLLARY: IN POLY TIME CAN DETERMINE IF A NORMAL SURFACE IS CONNECTED, AND ORIENTABLE.