A Connectivity Theory for Simplicial Complexes

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Goal: Construct a family of groups

$$A_n^q(\Delta, \sigma_0)$$

$n \geq 1$, $0 \leq q \leq \dim(\Delta)$, Δ simpl. complex
σ_0 maximal simplex of dimension $\geq q$.
similar to homotopy theory, but sensitive
to the combinatorics of Δ.

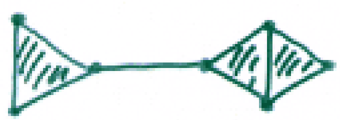
Origin: work of R. Atkin. Used
simpl. complexes to model
incidence relations in social systems.
(Q-analysis.)

Definitions. Δ simplicial complex, $0 \le q \le \dim \Delta$

1) $\sigma, \tau \in \Delta$, $\dim \ge q$. Then σ and τ are q-**connected** if there is a sequence

$$\sigma = \sigma_0 - \sigma_1 - \cdots - \sigma_n = \tau$$

with $|\sigma_i \cap \sigma_{i+1}| \ge q+1$ for all i.



2) Δ is q-**connected** if any two simplices of dim $\ge q$ are q-connected.

3) A q-**loop** in Δ is a q-connected sequence of simplices

$$\sigma_0 - \sigma_1 - \cdots - \sigma_n - \sigma_0$$

4) q-homotopy of q-loops: equivalence relation generated by:

i) $(\sigma_0 - \cdots - \sigma_i - \cdots - \sigma_0) \simeq_A (\sigma_0 - \cdots - \sigma_i - \sigma_i - \cdots - \sigma_0)$

ii)

$$\begin{array}{ccccccccccccc} \sigma_0 & - & \sigma_1 & - & \sigma_2 & - & \cdots & - & \sigma_n & - & \sigma_0 \\ \| & & | & & | & & \cdots & & | & & \| \\ \sigma_0 & - & \tau_1 & - & \tau_2 & - & \cdots & - & \tau_n & - & \sigma_0 \end{array}$$

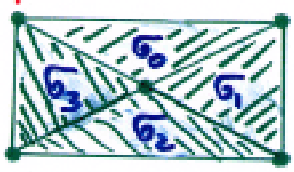
(each link represents q-connection)

5) $A_1^q(\Delta, \sigma_0)$ = set of all equivalence classes of q-loops under this relation.

Theorem. Concatenation of q-loops makes $A_1^q(\Delta, \sigma_0)$ into a group.

Examples.

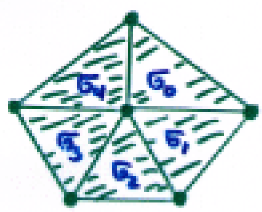
1)

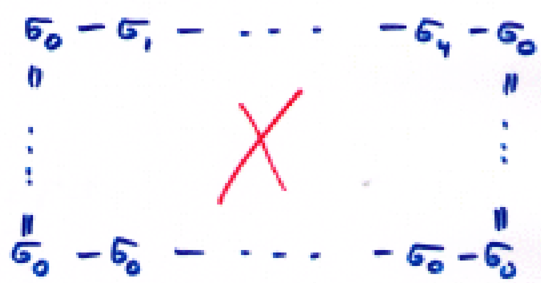


$$\begin{array}{ccccccccc}
\sigma_0 & - & \sigma_1 & - & \sigma_2 & - & \sigma_3 & - & \sigma_0 \\
\| & & | & & | & & | & & \| \\
\sigma_0 & - & \sigma_1 & - & \sigma_1 & - & \sigma_0 & - & \sigma_0 \\
\| & & | & & | & & | & & \| \\
\sigma_0 & - & \sigma_0 & - & \sigma_0 & - & \sigma_0 & - & \sigma_0
\end{array}$$

$$A_1'(\Delta, \sigma_0) = \{*\}$$

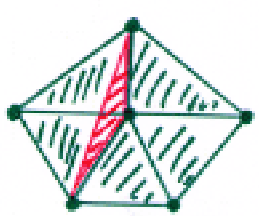
2)





$$A_1'(\Delta, \sigma_0) \cong \mathbb{Z}$$

3)



$$A_1'(\Delta, \sigma_0) = \{*\}$$

Note: Get a group

$$A_1^G(\Gamma, v_0)$$

for graphs Γ with distinguished vertex v_0.

Seifert - vonKampen Theorem.

Δ q-connected simplicial complex, Δ_1, Δ_2 q-connected subcomplexes s.t. $\Delta_1 \cup \Delta_2 = \Delta$, and disting. simplex $\sigma_0 \in \Delta_1 \cap \Delta_2$ q-connected. Furthermore, all q-loops with at most 4 simplices lie either in Δ_1 or Δ_2. Then

$$A_1^q(\Delta, \sigma_0) \cong A_1^q(\Delta_1, \sigma_0) * A_1^q(\Delta_2, \sigma_0) / \langle [\sigma] * [\sigma]^{-1} \rangle$$

$[\sigma]$ q-loop in $\Delta_1 \cap \Delta_2$.

Computation of $A_1^q(\Delta, \sigma_0)$.

Graph $\Gamma^q(\Delta, v_0)$:

vertices: simplices of Δ of dim $\geq q$

$(v_0 \leftrightarrow \sigma_0)$

edges: $v - w$ edge iff the corresp. simplices share a q-face

Space $X^q(\Delta, \sigma_0)$: glue 2-cells into all 3- and 4-cycles of $\Gamma^q(\Delta, v_0)$

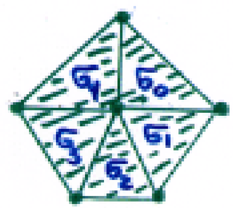
Theorem.

$$A_1^q(\Delta, \sigma_0) \cong \pi_1(\Gamma^q(\Delta, v_0)) / \langle 3\text{-}, 4\text{-cyles} \rangle$$

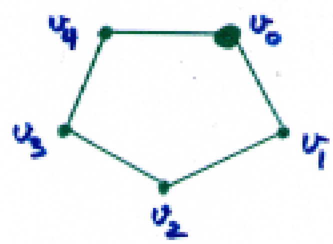
$$\cong \pi_1(X^q(\Delta, \sigma_0)).$$

Examples.

1)



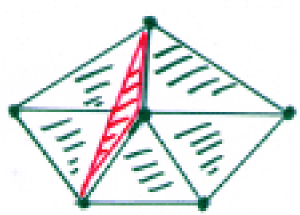
(Δ, σ_0)



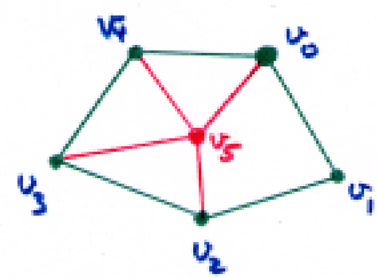
$\Gamma'(\Delta, v_0)$

$A_1'(\Delta, \sigma_0) \cong \mathbb{Z}$

2)



(Δ, σ_0)

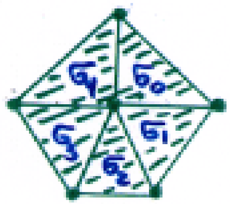


$\Gamma'(\Delta, v_0)$.

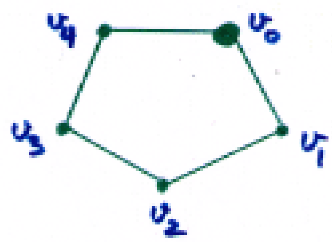
$A_1'(\Delta, \sigma_0) = \{*\}$

Examples.

1)



(Δ, σ_0)



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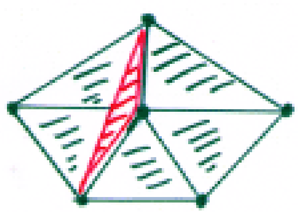
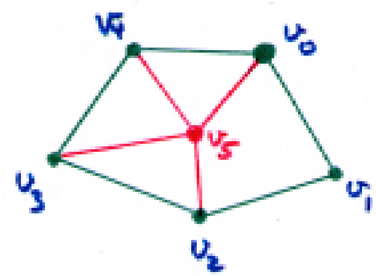
2)



(Δ, σ_0)



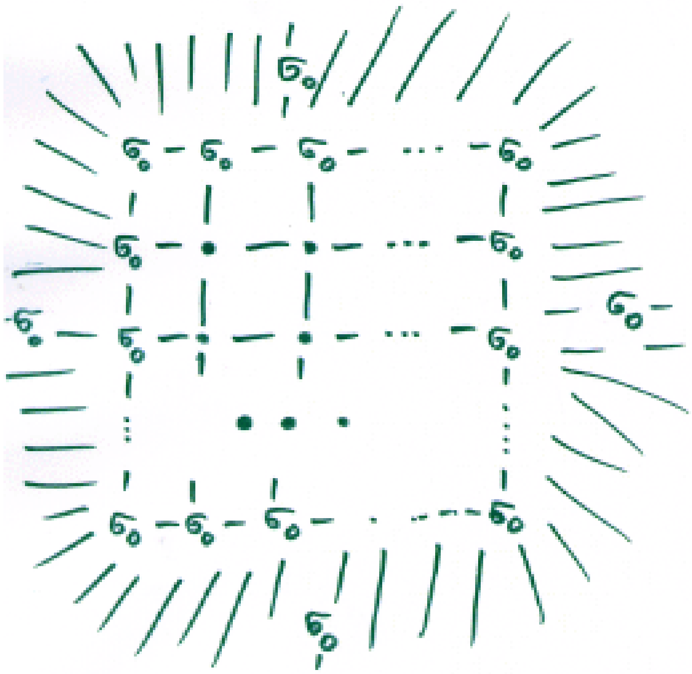
$\Gamma'(\Delta, v_0)$.

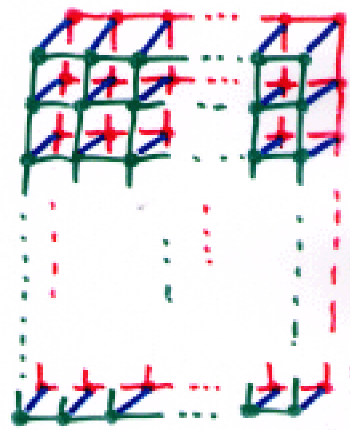
$A_1'(\Delta, \sigma_0) = \{*\}$

Higher dimensional groups $A_n^q(\Delta, \sigma_0)$, $n \geq 1$

Paradigm: $\pi_{n+1}(X) \cong \pi_n(\Omega X)$

$A_2^q(\Delta, \sigma_0)$:





infinite "grids"
almost all $= \sigma_0$

homotopy
equivalence

Theorem. $A_n^q(\Delta, \sigma_0)$ is abelian for $n \geq 2$.

Conjecture. There is a space $X^q(\Delta, \sigma_0)$, obtained from $\Gamma^q(\Delta, v_0)$ by attaching (infinitely many) cells, such that

$$A_n^q(\Delta, \sigma_0) \cong \pi_n(X^q(\Delta, \sigma_0), v_0).$$

Relative Theory. For $\sigma_0 \in \Sigma \subset \Delta$, have $A_n^q(\Delta, \Sigma, \sigma_0)$, $n \geq 1$

Theorem. There is a long exact sequence

$$\cdots \to A_n(\Sigma) \to A_n(\Delta) \to A_n(\Delta, \Sigma) \to A_{n-1}(\Sigma) \to \cdots$$

Let $\mathcal{C}_q$ be the category of simplicial complexes of dimension $\geq q$, with distinguished base simplex σ_0, $\dim(\sigma_0) \geq q$, and simplicial maps that do not collapse simplices. Then

$$A^q_*(-,-) : \mathcal{C}_q \longrightarrow \text{Groups}$$

is a graded functor.

Problem. Characterize complexes/graphs with trivial A-groups.

Booleau Lattices.

B_n = lattice of subsets of $\{1, 2, \ldots, n\}$

$\Delta(B_n)$ = order complex of B_n

$A_1^{n-3}(\Delta(B_n))^{ab} \cong \mathbb{Z}^r$

n	3	4	5	6
r	1	7	31	111

1, 7, 31, 111, ?

Theorem (Maurer) Δ matroid of dimension d. Then

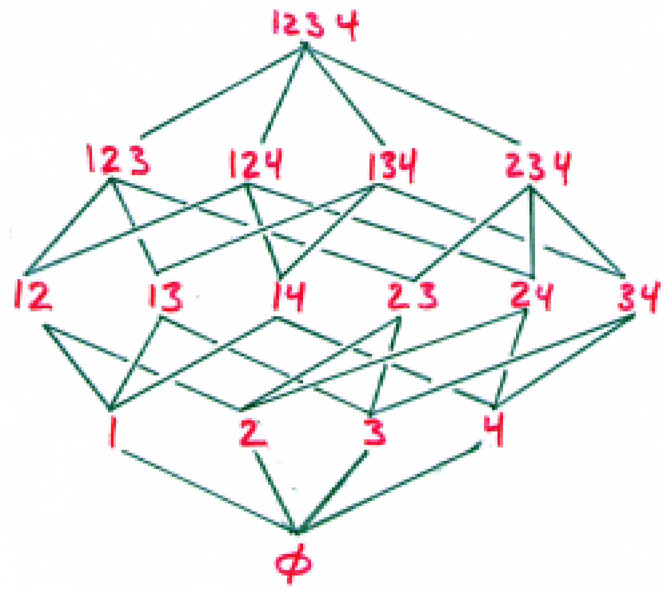
$$A_1^{d-1}(\Delta) = *.$$

Theorem (Lovász) G k-connected graph with n vertices, $v_1, \ldots, v_k$ vertices of G, $n_1, \ldots, n_k$ positive integers s.t. $n_1 + \cdots + n_k = n$. Then there exists a partition $V_1, \ldots, V_k$ of $V(G)$ s.t. $v_i \in V_i$, $|V_i| = n_i$, and each V_i spans a connected subgraph of G.

For $k=3$, attach 2-cells to 3- and 4-cyles in a graph made from spanning trees of G.

$A_1'(\Delta(\mathcal{B}_4))$

$\mathcal{B}_4$



$\Gamma'(\Delta(B_4))$

1- skeleton of the permutahedron

$X^1(\Delta(B_4))$

1- skeleton of the permutahedron

$$A_1^1(\Delta(B_4))^{ab} \cong \pi_1(X^1(\Delta(B_4)))^{ab} \cong \mathbb{Z}^7$$

Theorem. (Babson) Let $M_{3,n}^{\mathbb{R}}$ be the complement of the 3-equal arrangement in $\mathbb{R}^n$. Then $M_{3,n}^{\mathbb{R}}$ is homotopy equivalent to the space obtained by removing all faces of the permutahedron except for 0- and 1-dimensional faces and quadrangles.

Corollary. $A_1^{n-3}(\Delta(B_n)) \cong \pi_1(M_{3,n}^{\mathbb{R}})$, $n \geq 3$.

Theorem (Björner, Welker)
$H_1(M_{3,n}^{\mathbb{R}})$ is free of rank

$$2^{n-3}(n^2 - 5n + 8) - 1.$$

1, 7, 31, 111, 351, ...

Conjecture.

$$A_m^q(\Delta(B_n)) \cong \pi_m(M^{\mathbb{R}}_{n-q,n}), \quad m \geq 1$$

$\|$

0 for $m \geq 2$ ($k=3$)

(Khovanov)

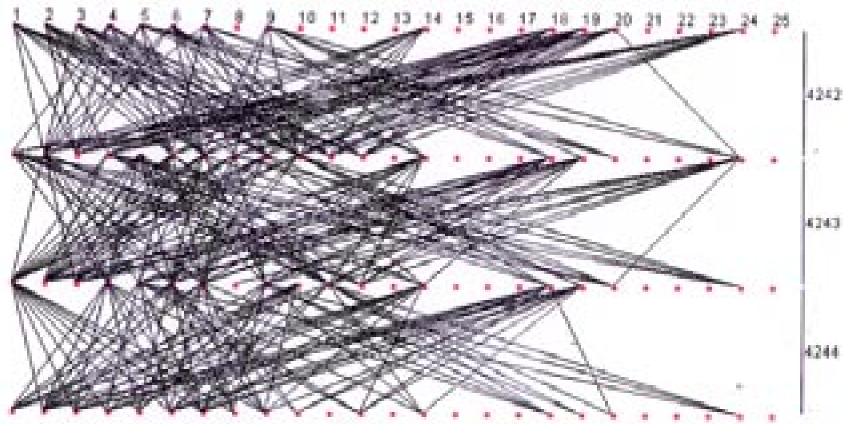
Analysis of Networks.

- "agents" $x_1, \ldots, x_n$
- "local" functions $f_1, \ldots, f_n$ which compute the state of $x_1, \ldots, x_n$
- each f_i depends on the states of (some of the x_i.

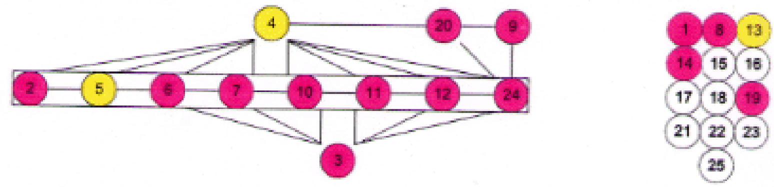
Goal. Make topological / combinatorial models of agent interaction and find suitable invariants.

- biological networks
- information networks
- social networks

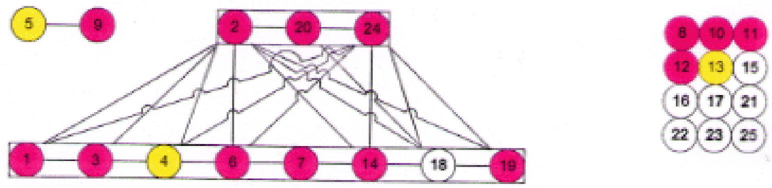
Interaction Poset for P=8 and T=4 , S=3



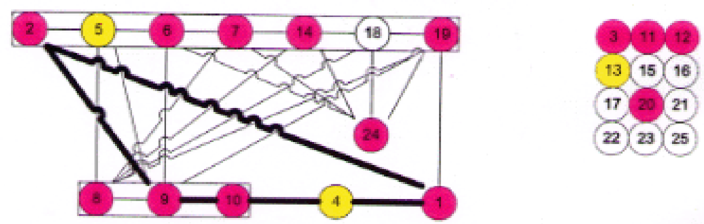
Time 4236



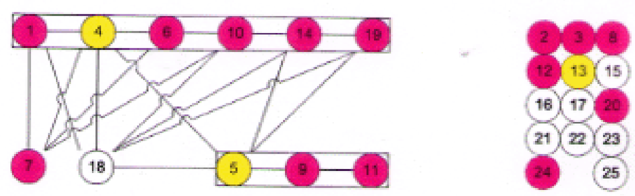
Time 4242



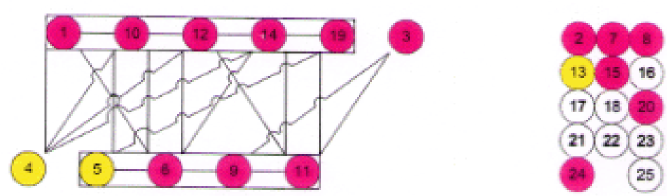
Time 4243



Time 4244



Time 4246



P=8,T=4,S=1,Q=4

○ High Fitness ● Average Fitness ● Low Fitness