Cell Complexes in Computational Algebraic Geometry

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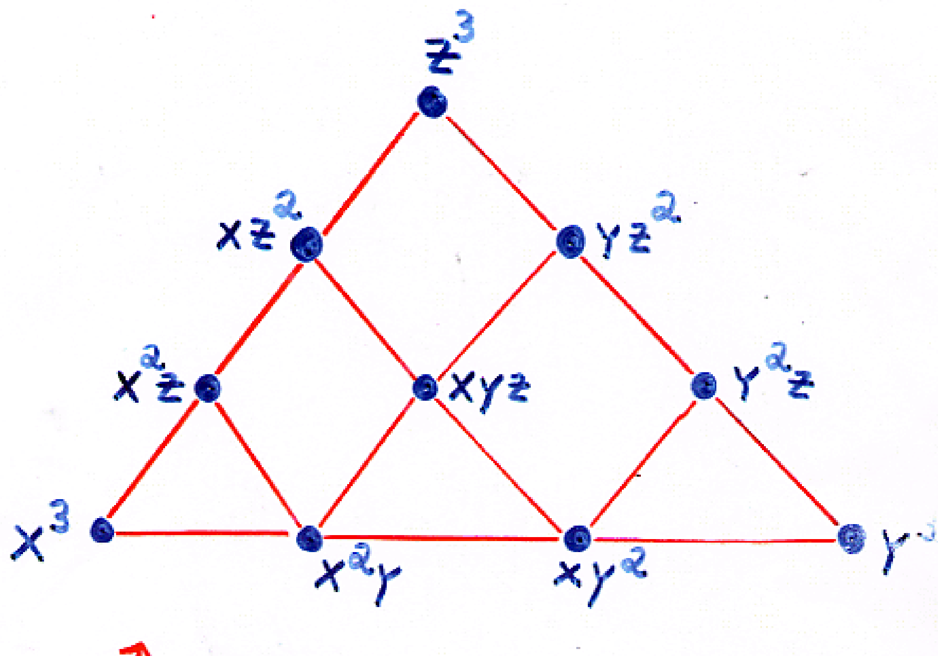


a cell complex (3 triangles, 3 quadrangles)

Algebraic Geometry

studies zero sets of polynomials

$$x^2y + 3xz^2 + 17z^3 \qquad \varphi$$

Gröbner Bases

reduce questions about polynomials to questions about monomials

$$x^2y$$

E.g. compute dimension, degree, ...

The Hilbert Series

Input: r monomials $m_1, \dots, m_r$ in n variables $x_1, \dots, x_n$

Output: the formal sum over all monomials not divisible by any m_i.

This a rational generating function of the form

$$\frac{K(x_1, \dots, x_n)}{(1-x_1)(1-x_2)\cdots(1-x_n)}$$

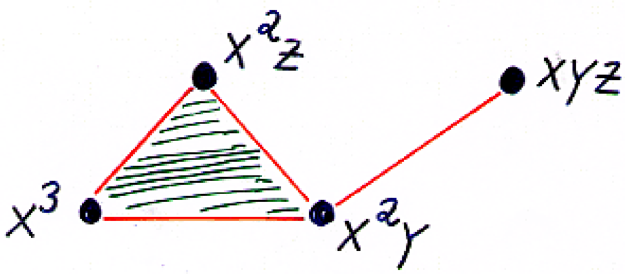
we want to compute this polynomial

Inclusion - Exclusion

$$K(x_1,\dots,x_n) = \sum_{I \subseteq \{1,\dots,r\}} (-1)^{|I|} \cdot m_I$$

where $\quad m_I = \text{l.c.m.}\{m_i : i \in I\}$

This formula is useless in practise.
Our task is to identify the few terms that survive the cancellations.

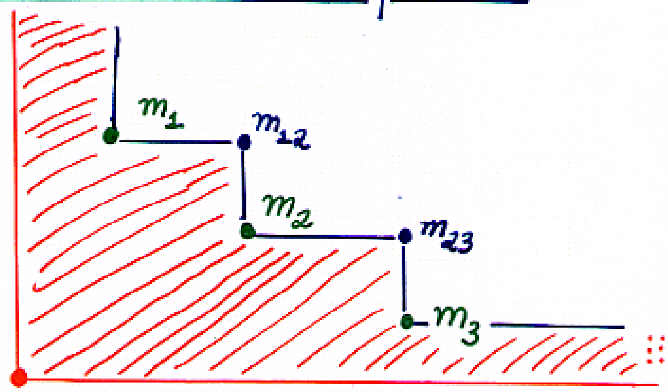


$n=3$
$r=4$

$$K(x,y,z) = 1 - x^3 - x^2z - x^2y - xyz$$
$$+ x^3y + x^3z + 2x^2yz - x^3yz$$

Staircase in the plane

$r = 3$
$n = 2$



$$K(x,y) = 1 - m_1 - m_2 - m_3 + m_{12} + m_{23}$$

Hilbert series = all monomials
− multiples of m_1, m_2 or m_3
+ common multiples of m_1 and m_2
+ common multiples of m_2 and m_3

Q: For $n = 2$ and general $r > 0$
how many terms does $K(x,y)$ have?

A: $2 \cdot r$

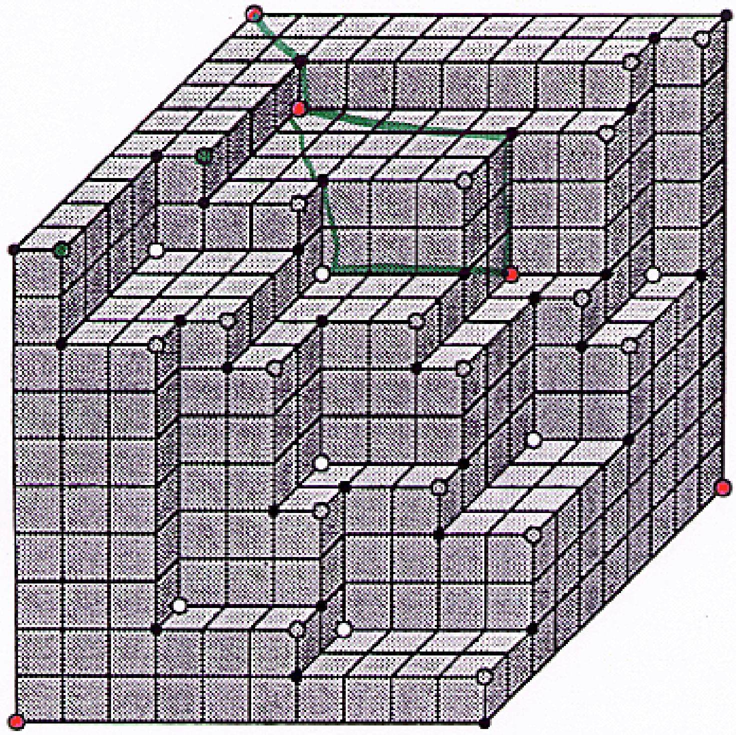
The Scarf Complex

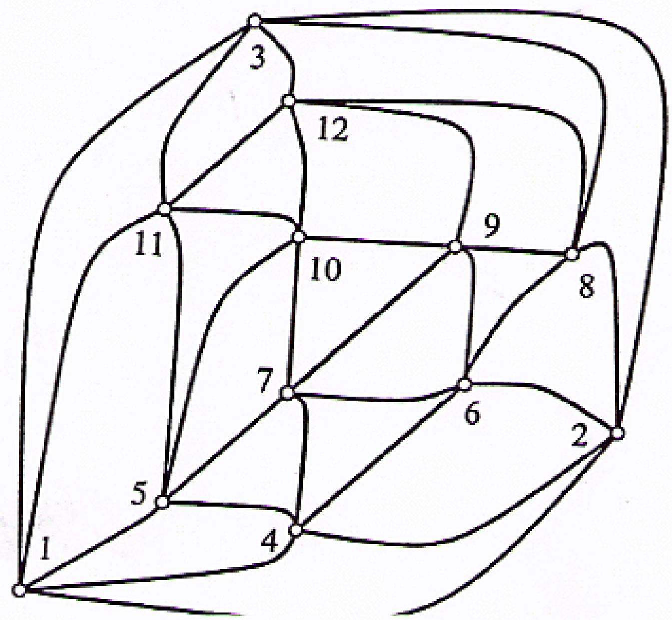
... is the simplicial complex Δ on $\{1,2,\ldots,r\}$ defined as follows:

I is a face of Δ iff there is no other subset J of $\{1,\ldots,r\}$ satisfying $m_I = m_J$.

Theorem: If $m_1,\ldots,m_r$ are generic (i.e. no variable x_i appears twice with the same exponent) then

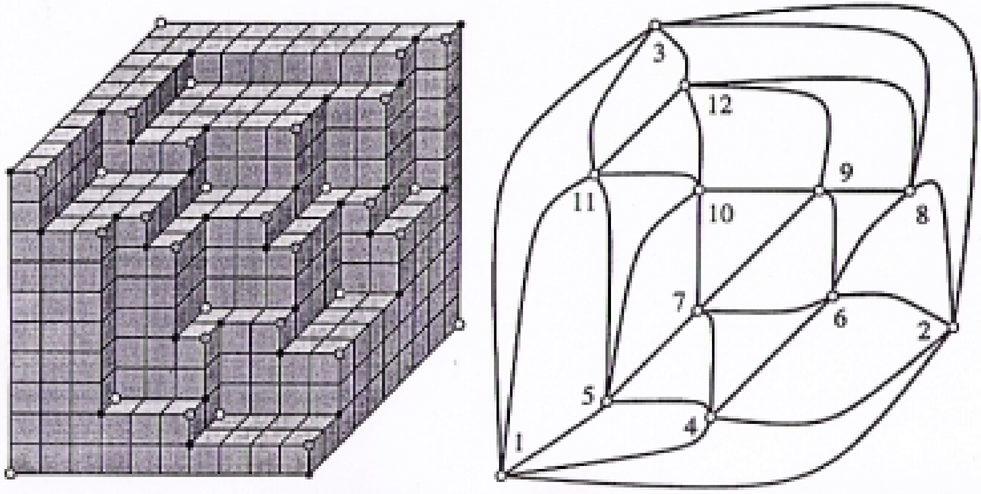
$$K(x_1,\ldots,x_n) = \sum_{I\in\Delta} (-1)^{|I|} \cdot m_I$$




3
12
11
9
10
8
7
6
2
5
4
1

$$
\begin{array}{cccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
\langle x^{10}, & y^{10}, & z^{10}, & x^8y^6z, & x^9y^3z^2, & x^4y^8z^3, & x^7y^5z^4, & xy^9z^5, & x^3y^7z^6, & x^5y^4z^7, & x^6yz^8, & x^2y^2z^9\rangle
\end{array}
$$

$$
\begin{aligned}
= \;& \langle x^{10}, y^{10}, z\rangle \cap \langle x^8, y^{10}, z^3\rangle \cap \langle x^7, y^8, z^6\rangle \cap \langle x^9, y^5, z^7\rangle \cap \langle x^7, y^7, z^7\rangle \cap \\
& \langle x^{10}, y^3, z^8\rangle \cap \langle x^9, y^4, z^8\rangle \cap \langle x^5, y^7, z^9\rangle \cap \langle x^3, y^9, z^9\rangle \cap \langle x^{10}, y, z^{10}\rangle \cap \\
& \langle x^2, y^9, z^{10}\rangle \cap \langle x, y^{10}, z^{10}\rangle \cap \langle x^8, y^8, z^4\rangle \cap \langle x^9, y^6, z^4\rangle \cap \langle x^4, y^{10}, z^5\rangle \cap \\
& \langle x^4, y^9, z^6\rangle \cap \langle x^6, y^4, z^9\rangle \cap \langle x^{10}, y^6, z^2\rangle \cap \langle x^6, y^2, z^{10}\rangle
\end{aligned}
$$



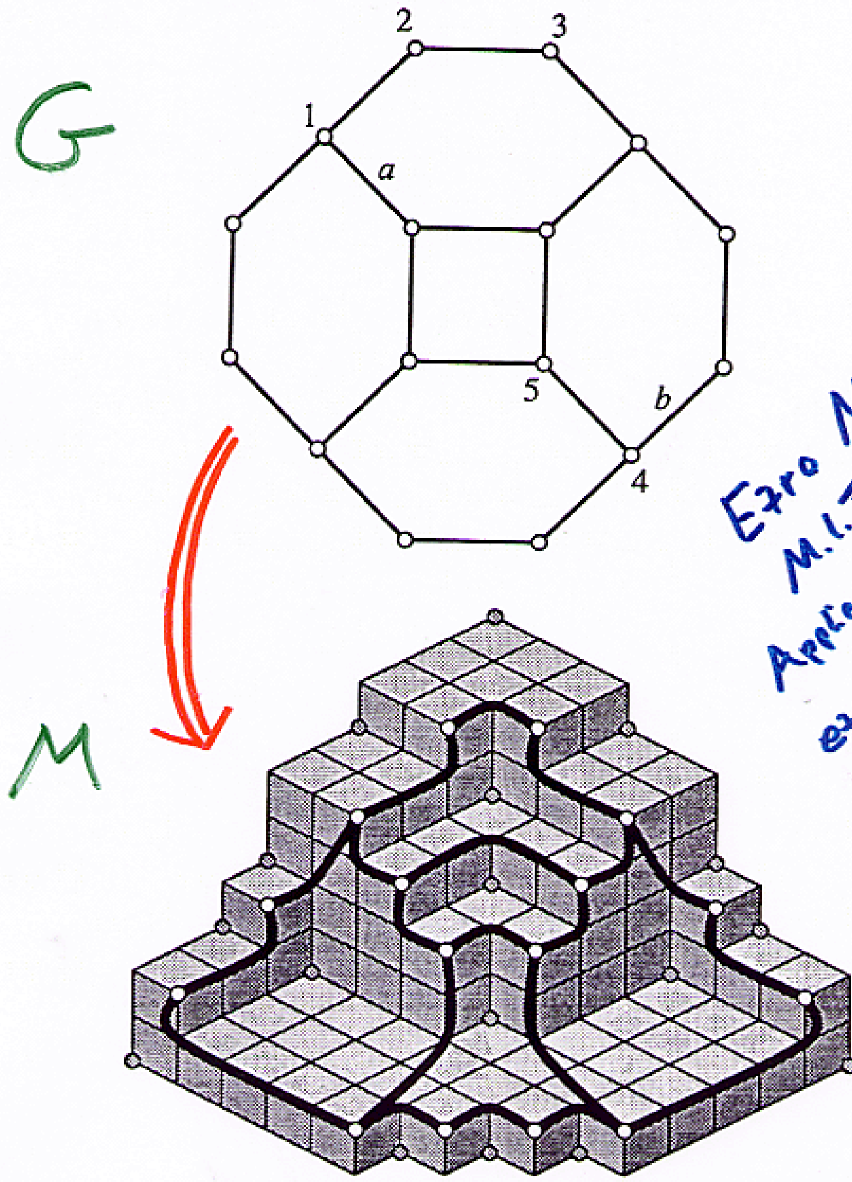
Three Variables

If $m_1, \ldots, m_r$ are monomials in x, y, z then there exists a planar map G with vertices $m_1, \ldots, m_r$ and edges labeled m_{ij} and regions labeled m_I such that $K(x,y,z)$ equals the graded Euler characteristic of G.

Conversely, every 3-connected planar graph G arises in this way.

[E. Miller 2001]

⇝ Steinitz' Theorem



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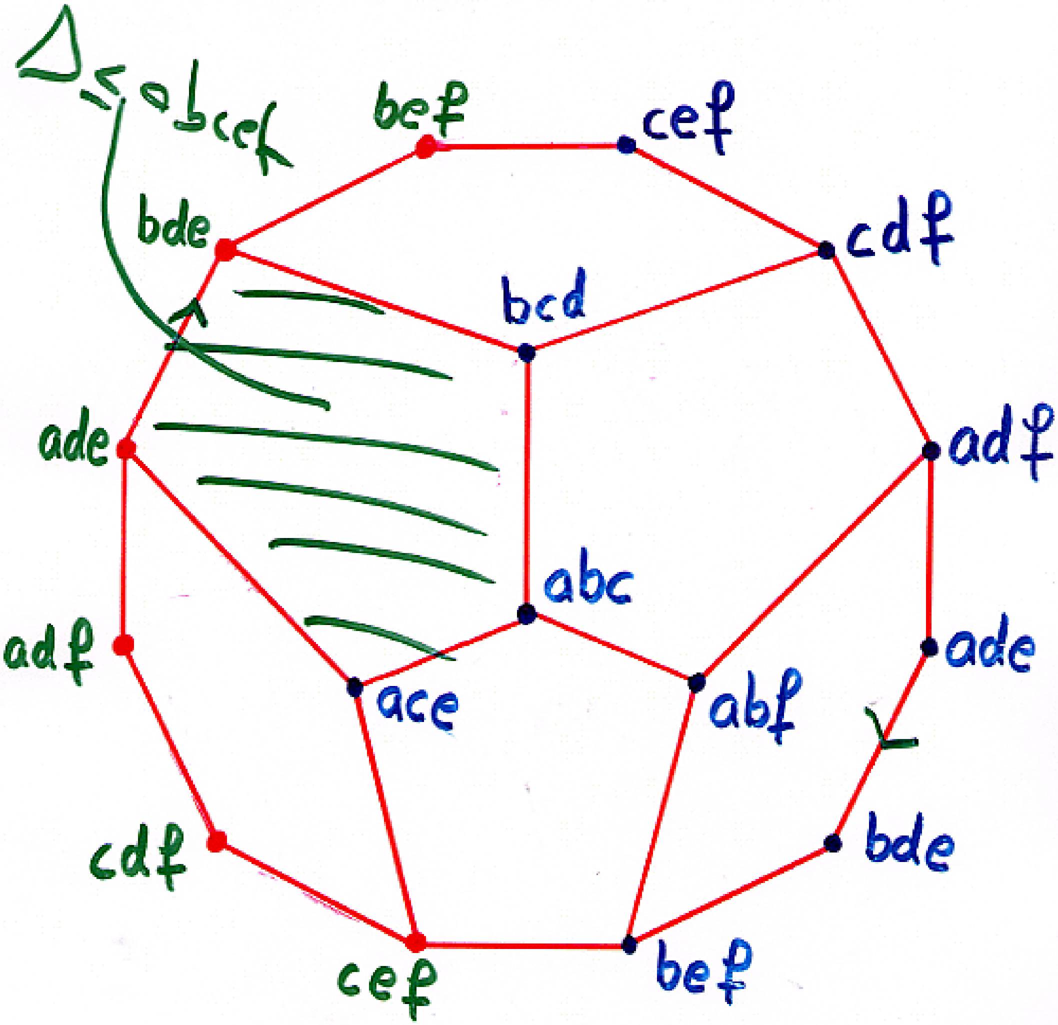
The Real Projective Plane



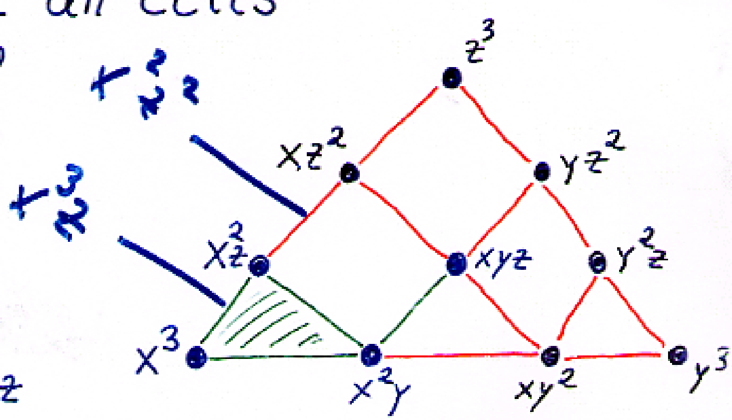
This is a cellular resolution
if and only if
the characteristic of $\mathbb{k}$ is not 2.

Labeled Cell Complexes

Δ a polyhedral cell complex with r vertices, labeled by monomials $m_1, m_2, \ldots, m_r$.

We label each cell of Δ by the least common multiple of the labels of the vertices on that cell.

For any other monomial m let $\Delta_{\leq m}$ be the subcomplex consisting of all cells whose label divides m.



$\Delta_{\leq x^3yz}$

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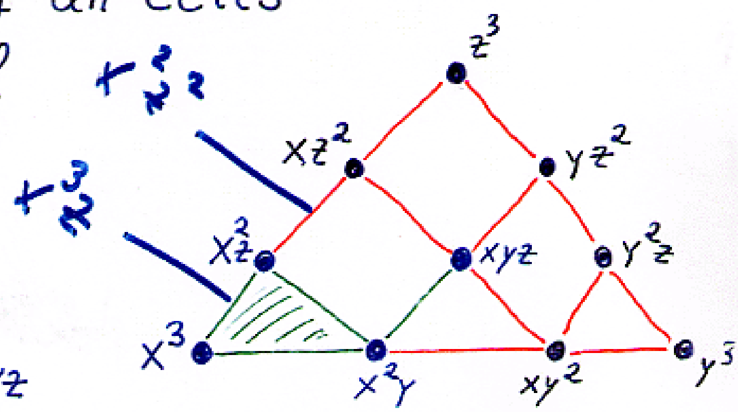
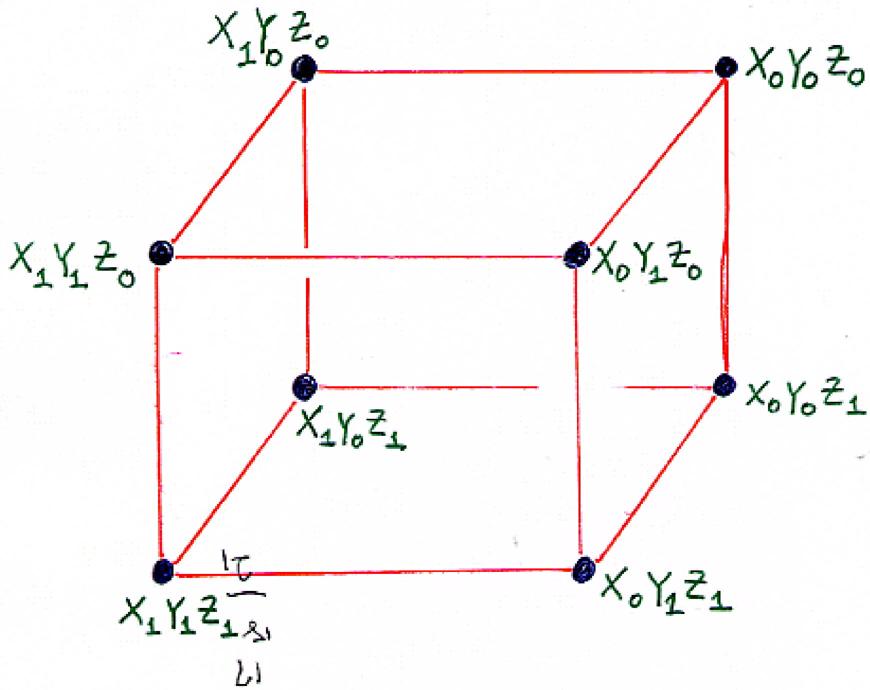


Every simple polytope supports a natural cellular resolution



$$M = \langle x_0, x_1\rangle \cap \langle y_0, y_1\rangle \cap \langle z_0, z_1\rangle$$

is the "irrelevant ideal" of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

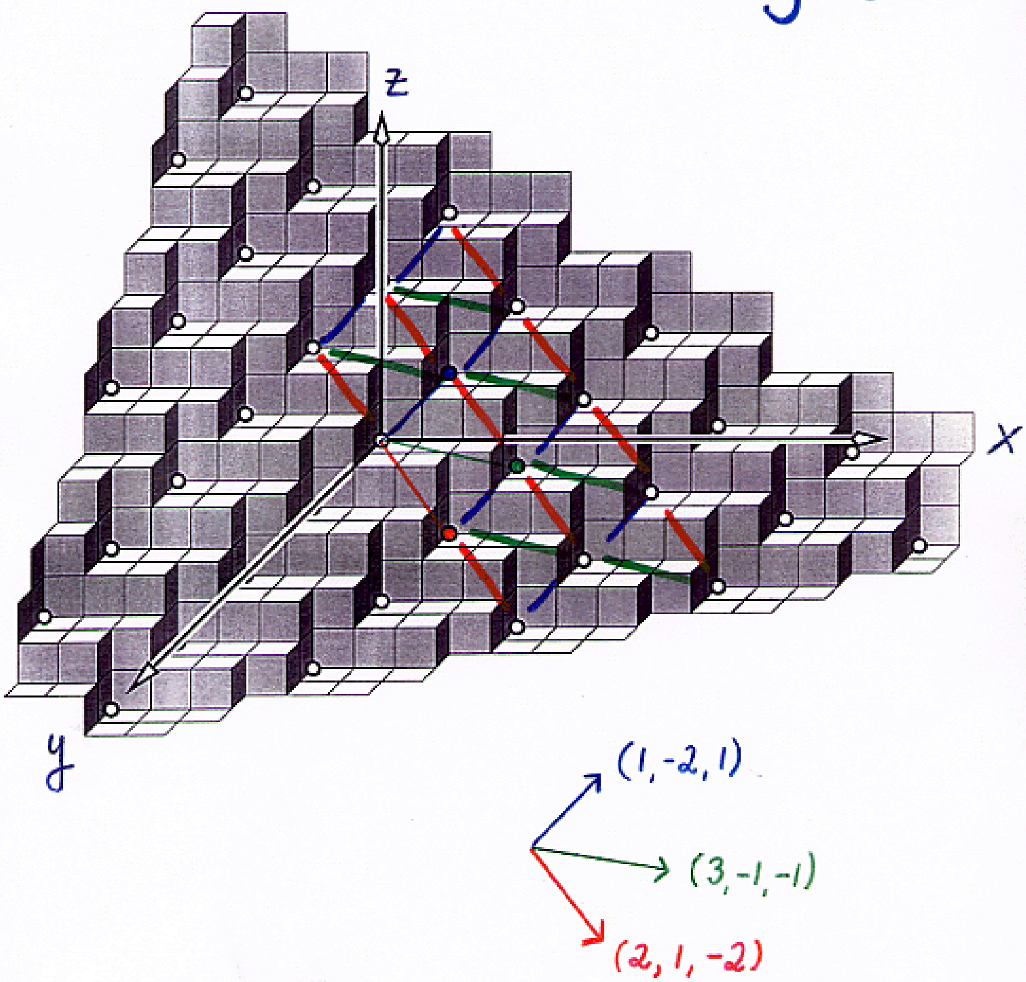
This works for any smooth toric variety ....

.... sheaf cohomology ...

A lattice module in $k[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$

$$M_{\mathcal{L}} = S \cdot \{ x^i y^j z^k : 3i + 4j + 5k = 0 \}$$

is generic



Important
future direction:

Study the
topology of
4-dimensional staircases,
5-dimensional staircases,
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